

Essays on Higher Order Approximation Solution Methods for DSGE Models

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von

Hong Lan, MSc

Präsident/Präsidentin der Humboldt-Universität zu Berlin:

Prof. Dr. Jan-Hendrik Olbertz

Dekan/Dekanin der Wirtschaftswissenschaftlichen Fakultät:

Prof. Dr. Ulrich Kamecke

Gutachter/Gutachterin:

1. Prof. Michael Burda, PhD

2. Prof. Lutz Weinke, PhD

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Chapter 1

Introduction

The reduction in the volatility of major macroeconomic time series such as GDP, consumption, employment and investment is a well-documented empirical regularity in postwar U.S. data (see Stock and Watson (2003) for a review). To study the origin of this decline in volatility, Justiniano and Primiceri (2008) and Fernández-Villaverde and Rubio-Ramírez (2007) initiate the use of dynamic, stochastic general equilibrium (DSGE) models with stochastic volatility: in a general equilibrium environment, model the macroeconomic aggregates of interest as the endogenous variables, and allow the standard deviation of the exogenous, structural innovations to change over time. This time-variation in the volatility of the exogenous innovations is implemented as shocks to the volatility of the exogenous innovations. Hence the name, volatility shock or stochastic volatility. While both of the two studies find that stochastic volatility in productivity and investment specific technology contribute significantly to the changes in the volatility of output, that how exactly stochastic volatility propagates through a general equilibrium is still an ongoing discussion. So is the quantitative impact of stochastic volatility on major macroeconomic aggregates.

This study aims to propose a propagation mechanism of stochastic volatility in a dynamic, stochastic general equilibrium, and to evaluate the quantitative effect of stochastic volatility on major macroeconomic aggregates in such an environment. In doing so, chapter 2 lays out a candidate DSGE model (a standard business cycle model with labor market search) that will be used for the analysis, and compares different solution methods for this model. While a third order standard state-space perturbation indeed produces a highly accurate approximation to the solution of the model, it is unable to produce stable impulse responses up to third order, and can be potentially explosive in simulation.

To restore the desirable stability in approximation, chapter 3 derives a novel per-

turbation method, the nonlinear moving average perturbation, that provides a direct mapping from a history of innovations to endogenous variables, decomposes the contributions from individual orders of uncertainty and nonlinearity, and enables familiar impulse response analysis in nonlinear settings. When the linear approximation is saddle stable and free of unit roots, higher order terms are likewise saddle stable. Chapter 4 further derives the state-space, recursive representation (recursion in the variables) of the nonlinear moving average perturbation up to third order, and shows this representation facilitates the calculation of the theoretical moments of model variables. Applying the nonlinear moving average perturbation method developed, chapter 4 analyzes the propagation mechanism of stochastic volatility in a business cycle model with labor market search, and studies the quantitative effect of stochastic volatility on major macroeconomic aggregates.

Chapter 2

Comparing Solution Methods for DSGE Models with Labor Market Search

Hong Lan*

Abstract

I compare the performance of solution methods in solving a standard real business cycle model with labor market search frictions. Under the conventional calibration, the model is solved by the projection method using the Chebyshev polynomials as its basis, and the perturbation methods up to third order in both levels and logs. Evaluated by two accuracy tests, the projection approximation achieves the highest degree of accuracy, closely followed by the third order perturbation in levels. Although different in accuracy, all the approximated solutions produce the simulated moments similar in value.

JEL classification: C63, C68, E32

Keywords: Computational methods, DSGE, Search and matching, Accuracy test

*Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Email: lanhong@cms.hu-berlin.de

2.1 Introduction

Initiated by Merz (1995) and Andolfatto (1996), many studies of business cycles choose to incorporate the search frictions introduced by Pissarides (1985) and Mortensen and Pissarides (1994) in their characterization of the labor market. While various methods are employed to solve this type of business cycle models with labor market search frictions, little effort has been made to compare the performance of these solution methods. I present a baseline model of this type, and solve it using projection and perturbation methods under the conventional calibration. Whereas the approximated solutions provided by these two classes of methods are different in accuracy, I find the simulated moments based on them are very similar in value.

The projection methods introduced by Judd (1992) have been shown to be able to produce a highly accurate approximation to the true policy function of a large class of DSGE models, and have therefore often been used as the reference solution of a model that has no known closed form solution, see Aruoba et al. (2006) and Caldara et al. (2012) for example. The projection I implement approximates the true solution of the model with a linear combination of the Chebyshev polynomials, and pins down the coefficients of the linear combination by minimizing a residual function derived from the Euler equations of the model at the nodes of the Chebyshev polynomials. The perturbation method introduced by Gaspar and Judd (1997) approximates the policy function with a Taylor expansion, and solves for the coefficients of the expansion from the equations resulting from successive differentiation of the equilibrium conditions of the model. With the perturbation method I approximate the policy rule up to third order for both the level and log specifications of the model. Then I implement Den Haan and Marcet's (1994) accuracy test and the Euler equation error test from Judd (1992) and Judd (1998) to evaluate the quality of the approximations produced by the two methods.

Of particular interest is that the equilibrium of the model is characterized by two intertemporal Euler equations. Besides the standard consumption Euler equation, employment is endogenously determined and also characterized by an intertemporal Euler equation. For each approximation and measured by the statistics of the two accuracy tests, I find that the consumption Euler equation is always better satisfied than the employment Euler equation. The projection approximation achieves the highest degree of accuracy in satisfying both of the two Euler equations, and the third order perturbation in levels is the second-best performing approximation. In particular, Den Haan and Marcet's (1994) test suggests that the first order perturbation in levels (the linear approximation) is superior to the first order perturbation in logs (log-linearization) in

satisfying the employment Euler equation. For the consumption Euler equation, the Euler equation error test suggests that the linear approximation performs better than the log-linearization, as noted by Aruoba et al. (2006) in their comparison of solution methods for a business cycle model where labor supply is also endogenously determined but characterized by an intratemporal Euler equation.

As above, the two accuracy tests complement each other in evaluating the approximations of this model. In practice, the Euler equation error test is often conducted on a state variable grid whose size is pre-specified merely with the guidance of the distributional properties of the state variables, without taking into account the correlation among the state variables implied by the corresponding approximation itself. As noted by Judd et al. (2010) and Judd et al. (2012), some regions on such a grid will not be visited in the equilibrium of the model.¹ Indeed, in this model, such redundant regions exist and the Euler equation errors computed in those regions are uninformative in evaluating the approximations. Den Haan and Marcet's (1994) test, however, builds up its test statistic using the simulated time series in which the correlation among the state variables as the restraint on the realizations has been enforced. Consequently, this test examines the accuracy of an approximation essentially in its associated state space where it ought to be accurate. One drawback of Den Haan and Marcet's (1994) test is that the results do not have an economic interpretation, but the results from the Euler equation error test do.

Although different in accuracy, all the approximations of this model produce similar simulated moments. This similarity follows from the fact that in the neighborhood of the deterministic steady state of the model, all the approximations behave similarly, and most of their realizations fall in that neighborhood in simulation. In recent literature, Petrosky-Nadeau and Zhang (2013) compare the performance of a spline approximation with the perturbation in logs up to second order in solving Hagedorn and Manovskii's (2008) model and find that the simulated moments produced by log-linearization is significantly different from those generated by the accurate spline approximation. Aside from that capital is not included, Hagedorn and Manovskii's (2008) model assumes a CES type of matching function, forcing the realization of the vacancy-filling rate to fall in between zero and one (see den Haan et al. (2000)). The model in this paper follows Merz (1995), Andolfatto (1996), Pissarides (2000), Shimer (2005), Pissarides (2009) and many others in assuming a standard Cobb-Douglas matching function, and interprets the vacancy-filling rate that exceeds

¹Instead of focusing on computing the Euler equation error, they make use of this observation to develop the projection methods on the realized (in simulation) state space only, for the purpose of mitigating the curse of dimensionality when solving models with a large number of state variables.

unity as, following Den Haan and De Wind (2012a), being due to firms hire more than one workers for a posted vacancy.

The rest of the paper is organized as follows. The real business cycle model with labor market search is specified in section 5.2. In section 2.3, I present the perturbation and projection approximations to the model with the calibration. The numerical results and implications of the approximations are analyzed in section 2.4. Section 5.6 concludes.

2.2 The Stochastic Growth Model with Labor Market Search

In this section, I lay out the model and characterize the equilibrium. The model embeds a Mortensen-Pissarides labor market search framework into an otherwise standard real business cycle model, and is parameterized close to the way described in Merz (1995) and Andolfatto (1996).

2.2.1 The model

The economy is populated by infinitely lived, identical households whose preferences are represented by the following utility function

$$U(c_t, n_t) = \ln c_t - \frac{n_t^{1-1/\gamma}}{1-1/\gamma} \quad (2.1)$$

where c_t is consumption, n_t the fraction of employed family members and γ the negative of the Frisch elasticity of labor supply. The model assumes only two states for a family member, employed or unemployed. The fraction of the unemployed family members therefore writes

$$u_t = 1 - n_t \quad (2.2)$$

Under appropriate assumptions on the matching function, the externality generated by labor market search activities can be internalized and therefore the model can be solved as a social planner's problem. The social planner evaluates the social welfare represented by the following value function

$$V(k_t, n_t, z_t) = \max_{c_t, v_t} \left\{ U(c_t, n_t) + \beta \mathbb{E}_t V(k_{t+1}, n_{t+1}, z_{t+1}) \right\} \quad (2.3)$$

where $\beta \in (0, 1)$ is the discount factor, k_t the capital stock, v_t the vacancy and z_t a stochastic productivity process of the form

$$z_t = \rho z_{t-1} + \varepsilon_{z,t}, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_z) \quad (2.4)$$

where $\rho \in (0, 1)$ is the persistence parameter of the process and ε_t the productivity shock, normally and identically distributed with zero mean and standard deviation σ_z . The maximization is subject to the following constraints

$$k_{t+1} = (1 - \delta)k_t + F(z_t, n_t, k_t) - c_t - \kappa_v v_t - \kappa_u (1 - n_t) \quad (2.5)$$

$$n_{t+1} = (1 - \chi)n_t + M((1 - n_t), v_t) \quad (2.6)$$

where (2.5) is the aggregate resource constraint with $\delta \in (0, 1)$ the depreciation rate of capital stock, κ_v the vacancy posting cost and κ_u the cost of job search, both assumed to be constant. $F(z_t, n_t, k_t)$ is the production function and assumed to take the Cobb-Douglas form

$$F(z_t, n_t, k_t) = e^{z_t} k_t^\alpha n_t^{1-\alpha} \quad (2.7)$$

where $\alpha \in (0, 1)$ is the capital share. The capital stock in the next period therefore is the sum of current capital after depreciation, and the current output net of consumption and two types of costs incurred by search and matching activities in labor markets.

The dynamic of aggregate employment is described by (2.6) with $\chi \in (0, 1)$ the exogenous job separation rate, assumed to be constant and $M((1 - n_t), v_t)$ the matching function. The employment in the next period therefore is the sum of current employment that has not been destroyed, and the new employment generated by the matching function. Following Merz (1995), Andolfatto (1996), Pissarides (2000), Shimer (2005), Pissarides (2009) and many others, the matching function is assumed Cobb-Douglas

$$M((1 - n_t), v_t) = m_0 v_t^{1-\eta} (1 - n_t)^\eta \quad (2.8)$$

where m_0 is a constant scaling factor and $\eta \in (0, 1)$ the elasticity of the matching function with respect to unemployment.

As is usual in labor market search and matching literature, the labor market tightness is defined as the ratio of the vacancy to the unemployment

$$\theta_t \equiv \frac{v_t}{1 - n_t} = \frac{v_t}{u_t} \quad (2.9)$$

The job finding rate is a function of the labor market tightness, measuring the rate at which unemployed workers find jobs, and is defined as the ratio of the job match to the unemployment

$$f_t \equiv \frac{M_t}{1 - n_t} = \frac{M_t}{u_t} \quad (2.10)$$

The vacancy filling rate is also a function of the labor market tightness, measuring the rate at which vacant jobs become filled, and is defined as the ratio of the job match to the vacancy

$$q_t \equiv \frac{M_t}{v_t} \quad (2.11)$$

Both the job finding and vacancy filling rate are probabilities, and should lie be-

tween zero and one. The vacancy filling rate, however, can potentially exceed unity in simulation when the matching function takes the Cobb-Douglas form (see den Haan et al. (2000, p. 485)). To avoid introducing nonsmoothness into the policy function since in that case the perturbation methods cannot be applied, I do not restrict q_t to be less than one. The realization of q_t that exceeds unity is interpreted as that firms hire more than one worker on each posted vacancy (see Den Haan and De Wind (2012a, p. 1480)).

2.2.2 Characterization

The equilibrium of the economy is characterized by, apart from the stochastic productivity process (2.4), the resource constraint (2.5) and aggregate employment dynamic (2.6), the Euler equation for consumption equalizing the expected present-discounted utility value of postponing consumption of one period to its utility value today

$$U_{c,t} = \beta \mathbb{E}_t \left[U_{c,t+1} \left(1 - \delta + F_{k,t+1} \right) \right] \quad (2.12)$$

where

$$U_{c,t} = \frac{1}{c_t} \quad (2.13)$$

$$F_{k,t} = \alpha e^{\tilde{z}_t} k_t^{\alpha-1} n_t^{1-\alpha} \quad (2.14)$$

and the Euler equation for employment equating the marginal loss in welfare due to vacancy creation, in terms of utility, to its expected present-discounted marginal contribution to social welfare

$$\frac{\kappa_v}{M_{v,t}} U_{c,t} = \beta \mathbb{E}_t \left[U_{c,t+1} \left(\frac{U_{n,t+1}}{U_{c,t+1}} + F_{n,t+1} + \kappa_u + \frac{\kappa_v}{M_{v,t+1}} \left(1 - \chi + M_{n,t+1} \right) \right) \right] \quad (2.15)$$

where

$$U_{n,t} = -n_t^{-1/\gamma} \quad (2.16)$$

$$F_{n,t} = (1 - \alpha) e^{\tilde{z}_t} k_t^\alpha n_t^{-\alpha} \quad (2.17)$$

and

$$M_{v,t} = (1 - \eta) m_0 v_t^{-\eta} (1 - n_t)^\eta \quad (2.18)$$

$$M_{n,t} = -\eta m_0 v_t^{1-\eta} (1 - n_t)^{\eta-1} \quad (2.19)$$

This marginal contribution, net of the disutility from work, is the sum of the marginal labor productivity, the saved job search cost and its potential continuation, i.e., in case the job match is not destroyed. $M_{n,t+1}$ corrects the continuation as the (un)employment stock has already been changed by the vacancy creation.

2.3 Solution Methods

The model described in section 5.2 does not have a known closed form solution, and needs to be solved with numerical methods. I solve the model using perturbation and a particular type of projection method, that is, the spectral method with Chebyshev polynomials.

The Perturbation method as described in Gaspar and Judd (1997), Judd and Guu (1997), Judd (1998, ch. 13), Jin and Judd (2002), Schmitt-Grohé and Uribe (2004) and many others, assumes the policy function exists, then successively differentiates the equilibrium conditions and solve the resulting system of equations evaluated at typically the deterministic steady state to recover the coefficients of a Taylor expansion of a desired order of the policy function. Under appropriate smoothness assumptions,² Taylor's theorem guarantees the expansion converges to the true policy function as the order of expansion approaches infinity.

The spectral method specifies the approximated policy function as a linear combination of Chebyshev polynomial basis, as noted in Judd (1992, p. 421), imposing smoothness conditions on the approximated policy function, and then solves for the coefficients of the linear combination by minimizing the residual function defined by the equilibrium conditions of the model at the chosen collocation points, i.e., the zeros of the Chebyshev polynomial basis. As noted in Aruoba et al. (2006, p. 2488), such a minimization process delivers the best trade-off between accuracy and the ability of handling a large number of basis functions, and by the Chebyshev interpolation theorem, the approximation error becomes arbitrarily small as the number of collocation points used in approximation approaches infinity.

2.3.1 Perturbation

The equilibrium conditions of the model, that is, (2.4)-(2.6) and the two Euler equations (5.23) and (5.24) can be cast into the following problem

$$0 = \mathbb{E}_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)] \quad (2.20)$$

where the n_y -dimensional vector-valued function $f : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}^{n_y}$ is assumed C^M with respect to all its arguments, where M is the order of approximation to be introduced subsequently; $y_t \in \mathbb{R}^{n_y}$ is the vector of n_y endogenous and exogenous

²I.e., all involved functionals are continuous and differentiable to at least the order of approximation, see Judd (1998, ch. 13) for example.

variables; and $\varepsilon_t \in \mathbb{R}^{n_e}$ the vector of n_e exogenous shocks,³ where n_y and n_e are positive integers ($n_y, n_e \in \mathbb{N}$). The elements of ε_t are assumed i.i.d. with $E[\varepsilon_t] = 0$ and $E[\varepsilon_t^{\otimes [m]}]$ finite $\forall m \leq M$.⁴

Following standard practice in DSGE perturbation, I introduce an auxiliary parameter $\sigma \in \mathbb{R}$ to scale the risk in the model.⁵ The stochastic model under study in (5.30) corresponds to $\sigma = 1$ and $\sigma = 0$ represents the deterministic version of the model. Indexing solutions with σ

$$y_t = y(\sigma, z_t), \quad y : \mathbb{R} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_y} \quad (2.21)$$

with the state vector z_t given by⁶

$$z_t = \begin{bmatrix} y_{t-1} \\ \varepsilon_t \end{bmatrix} \in \mathbb{R}^{n_z \times 1}, \quad \text{where } n_z = n_y + n_e \quad (2.22)$$

To enable a standard DSGE perturbation, I assume the vector function y exists and is C^M with respect to all its arguments. Time invariance of the policy function and scaling risk imply

$$y_{t+1} = \tilde{y}(\sigma, \tilde{z}_{t+1}), \quad \tilde{z}_{t+1} = \begin{bmatrix} y_t \\ \sigma \varepsilon_{t+1} \end{bmatrix} \in \mathbb{R}^{n_z \times 1}, \quad \tilde{y} : \mathbb{R} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_y} \quad (2.23)$$

The notation, y and \tilde{y} , is adopted to track the source (through y_t or y_{t+1}) of derivatives of the policy function. This is necessary as (i) the \tilde{z}_{t+1} argument of \tilde{y} is itself a function of y through its dependence on y_t , and (ii) σ scales ε_{t+1} in the \tilde{z}_{t+1} argument of \tilde{y} , but not ε_t in the z_t argument of y . This follows from the conditional expectations in (5.30): ε_t realizes at time t and is in the time t information set—hence, it is not scaled by σ ; however, ε_{t+1} has not yet been realized and is the source of risk—hence, it is scaled by σ .⁷

Inserting the policy functions for y_t and y_{t+1} —equations (3.2) and (3.4)—into

³Nonlinearity or serial correlation in exogenous processes can be captured by including the processes themselves in the vector y_t and including functions in f that specify the nonlinearity or correlation pattern.

⁴The notation $\varepsilon_t^{\otimes [m]}$ represents Kronecker powers. $\varepsilon_t^{\otimes [m]}$ is the m 'th fold Kronecker product of ε_t with itself: $\underbrace{\varepsilon_t \otimes \varepsilon_t \cdots \varepsilon_t}_{m \text{ times}}$.

⁵This formulation follows Adjemian et al.'s (2011) Dynare, Anderson et al.'s (2006) Perturbation-AIM and Juillard (2011). Jin and Judd's (2002) or Schmitt-Grohé and Uribe's (2004) model classes can be rearranged to fit (5.30).

⁶Only in this section, i.e., section 2.3.1, z_t is used to denote the state vector of the policy function. Everywhere, z_t denotes the productivity process.

⁷See also Anderson et al. (2006) and Juillard (2011) for similar discussions.

(5.30) yields

$$0 = E_t \left[f \left(\tilde{y} \left(\sigma, \begin{bmatrix} y(\sigma, z_t) \\ \sigma \varepsilon_{t+1} \end{bmatrix} \right), y(\sigma, z_t), z_t \right) \right] = F(\sigma, z_t) \quad (2.24)$$

a function with arguments σ and z_t .⁸ I will construct a Taylor series approximation of the solution (3.2) around a deterministic steady state defined as follows

Definition 2.3.1. Deterministic Steady State

Let $\bar{y} \in \mathbb{R}^{n_y}$ be a vector that solves the policy function (3.2) with $\varepsilon_t = 0$ and $\sigma = 0$

$$\bar{y} = y(0, \bar{z}), \text{ where } \bar{z} = \begin{bmatrix} \bar{y} \\ 0 \end{bmatrix} \quad (2.25)$$

In practice, the deterministic steady state value is solved from the deterministic version of (3.15), i.e., from $0 = f(\bar{y}, \bar{y}, \bar{z})$.

With f and y both being vector-valued functions that take vectors as arguments, their partial derivatives form hypercubes. I use the method of Lan and Meyer-Gohde (2013a) that differentiates conformably with the Kronecker product, allowing me to maintain standard linear algebraic structures to derive my results.

Definition 2.3.2. Matrix Derivatives

Let $A(B) : \mathbb{R}^{s \times 1} \rightarrow \mathbb{R}^{p \times q}$ be a matrix-valued function that maps an $s \times 1$ vector B into an $p \times q$ matrix $A(B)$, the derivative structure of $A(B)$ with respect to B is defined as

$$A_B \equiv \mathcal{D}_{B^T} \{A\} \equiv \begin{bmatrix} \frac{\partial}{\partial b_1} & \dots & \frac{\partial}{\partial b_s} \end{bmatrix} \otimes A \quad (2.26)$$

where b_i denotes i 'th row of vector B , T indicates transposition; n 'th derivatives are

$$A_{B^n} \equiv \mathcal{D}_{(B^T)^n} \{A\} \equiv \left(\begin{bmatrix} \frac{\partial}{\partial b_1} & \dots & \frac{\partial}{\partial b_s} \end{bmatrix}^{\otimes [n]} \right) \otimes A \quad (2.27)$$

I assume the policy function, (3.2), admits a Taylor series approximation up to M 'th order at a deterministic steady state which I write as⁹

$$y_t \approx \sum_{j=0}^M \frac{1}{j!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j \sigma^i} \sigma^i \right] (z_t - \bar{z})^{\otimes [j]} \quad (2.28)$$

⁸Note that ε_{t+1} is not an argument of F as it is the variable of integration inside the expectations. I.e.,

$$F(\sigma, z_t) = \int_{\Omega} f \left(\tilde{y} \left(\sigma, [y(\sigma, z_t) \sigma \varepsilon_{t+1}] \right), y(\sigma, z_t), z_t \right) \phi(\varepsilon_{t+1}) d\varepsilon_{t+1}$$

where Ω is the support and ϕ the p.d.f. of ε_{t+1} . Thus, when $\sigma = 0$, ε_{t+1} is no longer an argument of f and the integral (and hence the expectations operator) is superfluous, yielding the deterministic version of the model.

⁹See appendix A.0.1 for a derivation of the Taylor series approximation.

where $y_{z^j \sigma^i} \in \mathbb{R}^{n_y \times n_z^j}$ is the partial derivative of the vector function y with respect to the state vector z_t j times and the perturbation parameter σ i times evaluated at the deterministic steady state using the notation of definition 3.2.2. That is

$$\begin{aligned} y_{z^j \sigma^i} &\equiv \mathcal{D}_{z_{t-1}^T \sigma^i}^{j+i} \{y(\sigma, z_t)\} \equiv \left(\left[\frac{\partial}{\partial z_{1,t-1}} \quad \cdots \quad \frac{\partial}{\partial z_{n_z,t-1}} \right]^{\otimes [j]} \otimes \left(\frac{\partial}{\partial \sigma} \right)^{\otimes [i]} \right) \otimes y(\sigma, z_t) \\ &= \left(\left[\frac{\partial}{\partial z_{1,t-1}} \quad \cdots \quad \frac{\partial}{\partial z_{n_z,t-1}} \right]^{\otimes [j]} \left(\frac{\partial}{\partial \sigma} \right)^i \right) \otimes y(\sigma, z_t) \end{aligned} \quad (2.29)$$

where T indicates transposition and the second line follows as σ is a scalar. The terms $\left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j \sigma^i} \sigma^i \right]$ in (2.28) collect all the coefficients associated with the j 'th fold Kronecker product of the state vector, $(z_t - \bar{z})$. Higher orders of σ correct the Taylor series coefficients for risk by successively opening the coefficients to higher moments in the distribution of future shocks.¹⁰ At third order and for $\sigma = 1$, the Taylor approximation (2.28) writes

$$y_t \approx \bar{y} + \frac{1}{2} y_{\sigma^2} + \frac{1}{6} y_{\sigma^3} + \left[y_z + \frac{1}{2} y_{\sigma^2 z} \right] (z_t - \bar{z}) + \frac{1}{2} y_{z^2} (z_t - \bar{z})^{\otimes [2]} + \frac{1}{6} y_{z^3} (z_t - \bar{z})^{\otimes [3]} = \hat{y}_t \quad (2.30)$$

where only terms with nonzero coefficients have been included and $\hat{\cdot}$ highlights that (2.30) is an approximation of the policy function (3.2). To solve for the coefficients of the third order expansion (2.30), I take the collection of derivatives of f in (3.15) from the previous order (for the first order, I start with f itself) and

1. differentiate the derivatives of f from the previous order with respect to all their arguments
2. evaluate the partial derivatives of f and of y at the deterministic steady state
3. apply the expectations operator and evaluate using the given moments
4. set the resulting expression to zero and solve for the unknown partial derivatives of y .

The resulting equation for y_z at first order takes the form of a matrix quadratic.¹¹ All the other unknown coefficients, as noted by Judd (1998, ch. 13), Jin and Judd (2002), Schmitt-Grohé and Uribe (2004) and others, are solutions to linear equations taking the results from lower orders as given.¹²

¹⁰A similar interpretation can be found in Judd and Mertens (2013a) for univariate expansions and in Lan and Meyer-Gohde (2013a) for expansions in infinite sequences of innovations.

¹¹See, Uhlig (1999) for example.

¹²All these linear equations can be cast into a generalized Sylvester form, see Lan and Meyer-Gohde

2.3.2 Projection

The spectral method seeks an approximation of the policy function on the grid of state variables. The lower and upper bounds of this grid are chosen such that, as noted in Aruoba et al. (2006, p. 2486) and Caldara et al. (2012, p. 196), they will bind only with an extremely low probability. The deterministic steady state as given in definition 2.3.1 of the state variables is also included in the grid as it is a point that can be determined before approximation, see Judd (1992, p. 429). Given there are three state variables in the model, i.e., capital, employment and productivity, the grid of approximation is a cube, $[k_{min}, k_{max}] \times [n_{min}, n_{max}] \times [z_{min}, z_{max}]$ where the subscripts min and max indicate the lower and upper bounds of the state variables they attach to. Along each of the three dimensions, the grid points are chosen to be, up to a linear transformation, the roots of Chebyshev polynomials that lie in the interval between -1 and 1 .

The two policy functions of consumption and vacancy are both functions of state variables and are approximated with the following linear combination of the Chebyshev basis

$$\hat{c}_t = X(k_t, n_t, z_t) \Theta_c \quad (2.31)$$

$$\hat{v}_t = X(k_t, n_t, z_t) \Theta_v \quad (2.32)$$

where $\hat{\cdot}$ indicates these are approximations. Θ_c and Θ_v are two vectors of coefficients to be determined. Both \hat{c}_t and \hat{v}_t are of dimension $(n_g \times 1)$ with n_g the number of grid points. The multidimensional Chebyshev polynomial basis $X(k_t, n_t, z_t)$ on which the approximation of both consumption and vacancy are built is the Kronecker tensor product of three Chebyshev polynomial basis of capital, employment and productivity respectively. The details of constructing $X(k_t, n_t, z_t)$ are relegated to the appendix.

The two Euler equations (5.23) and (5.24) that characterize the policy function of consumption and vacancy can be written as the following functional

$$\mathcal{N}(c_t, v_t) = \begin{bmatrix} U_{c,t} - \beta \mathbb{E}_t \left[U_{c,t+1} \left(1 - \delta + F_{k,t+1} \right) \right] \\ \frac{\kappa_v}{M_{v,t}} U_{c,t} - \beta \mathbb{E}_t \left[U_{c,t+1} \left(\frac{U_{n,t+1}}{U_{c,t+1}} + F_{n,t+1} + \kappa_u + \frac{\kappa_v}{M_{v,t+1}} \left(1 - \chi + M_{n,t+1} \right) \right) \right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.33)$$

Inserting the approximated policy functions (2.31) and (2.32) in the previous functional, noting that k_{t+1} and n_{t+1} can be calculated using the aggregate resource constraint (2.5) and the dynamic of aggregate employment (2.6) given the state variable grid and approximated policy function, and approximating the expectation with, following Judd (1992), Gauss-Hermite quadrature method yields the residual function.

(2014).

The unknown coefficients of the approximated policy function, Θ_c and Θ_v , are solved from the residual function using den Haan and Marcet's (1990) functional iteration, taken the third order perturbation in levels as the initial guess. See the appendix for details.

2.3.3 Calibration

The model is quarterly calibrated. The parameter values as summarized in Table 2.1, are taken from Merz (1995), Petrongolo and Pissarides (2001), Shimer (2005) and Pissarides (2009). In particular, the steady state values of the labor market tightness

Table 2.1 Quarterly Calibration

symbol	value	symbol	value
γ	-1.25	χ	0.036
α	0.36	η	0.5
δ	0.026	θ_{ss}	0.72
β	0.99	n_{ss}	0.94
ρ	0.95	κ_v	0.0875
σ_z	0.0073	κ_u	0.1451

and aggregate employment, θ_{ss} and n_{ss} respectively, are taken from Shimer (2005) and Pissarides (2009). The vacancy posting cost, κ_v , is chosen, using the projection approximation, such that the standard deviation of vacancy relative to that of output is equal to 7.31 as reported in Merz (1995).¹³ Then solving the model in steady state pins down κ_u , the cost of job searching.

2.4 Numerical Results

This section first reports the simulated moments of the model using the projection approximation which will be shown as the top performing one among all the approximations considered in this paper. Such set of moments reveals the model's ability to replicate some of the key regularities of the business cycle and in particular, of the labor markets. Second, the simulated density of all the approximations will be presented. Third, the quality of the approximations will be examined by implementing Den Haan

¹³The projection solution is used to calibrate the model as it is most accurate approximation of the policy function evaluated with Den Haan and Marcet's (1994) accuracy test and the Euler equation error test. The detailed discussion of accuracy is presented in the next section.

and Marcet's (1994) test and the Euler equation error test from Judd (1992) and Judd (1998). Given the difference in accuracy among all the approximations and to study the implications of such difference, the simulated moments of all the approximations will be computed for comparison.

2.4.1 Simulated Moments

The model is simulated using the approximation generated by the projection method. This approximation outperforms all the perturbations in terms of accuracy. To this end, it is chosen as the benchmark that represents the model's ability of explaining the observed aggregate fluctuations, in particular the fluctuations of the labor market variables as they reflect the contribution of the search and matching framework incorporated in the model.

The simulation environment is similar to that specified in Merz (1995), Shimer (2005) and Petrosky-Nadeau and Zhang (2013): the model is simulated 1000 times. Each simulation contains 412 observations with the first 200 discarded. As the model is quarterly calibrated, each simulation contains effectively the observations of 212 quarters, corresponding to about 53 years of quarterly data presented in Shimer (2005) and Pissarides (2009). As the projection method approximates the model in levels, the simulated time series are transformed by the natural logarithm, and then detrended using the Hodrick-Prescott filter with a quarterly smoothing parameter 1600. From the 1000 simulations there are 1000 sets of moments, and only the average of these simulated moments is reported.

Table 2.2 Second moments from Data and Projection Solution

Statistic	Data	Model	Statistic	Data	Model
σ_c/σ_y	0.40	0.34	σ_u/σ_y	6.11	3.37
σ_k/σ_y	0.22	0.29	σ_v/σ_y	7.31	7.31
σ_n/σ_y	0.54	0.22	σ_θ/σ_p	19.10	10.32
σ_y	1.87	1.05	σ_p/σ_y	0.68	0.84
			$\rho(u, v)$	-0.894	-0.1957

The model performs well in generating relative volatilities in frequently reported business cycle aggregates such as consumption and capital stock. Along the labor market dimension, the volatility of labor market tightness relative to that of the labor productivity, σ_θ/σ_p , reaches 10.33. Whereas it is about half of 19.10 reported by

Shimer (2005), it already exceeds 7.56, a plausible target of a model with constant job destruction and productivity shock only (see Pissarides (2009)). Moreover, the model is capable of replicating the negatively sloped Beveridge curve, i.e., $\rho(u, v)$ in table 2.2. This is because that the aggregate unemployment as a state variable will not immediately respond to an increase in vacancy creating activities induced by a positive productivity shock. The household therefore cannot send more family members to searching which will lead to an increase in unemployment and a positive relationship between vacancy and unemployment. Given that the model assumes constant vacancy posting and searching cost, incorporating no frictions other than search, a richer structure is needed to generate an $\rho(u, v)$ that closer to the empirical target.

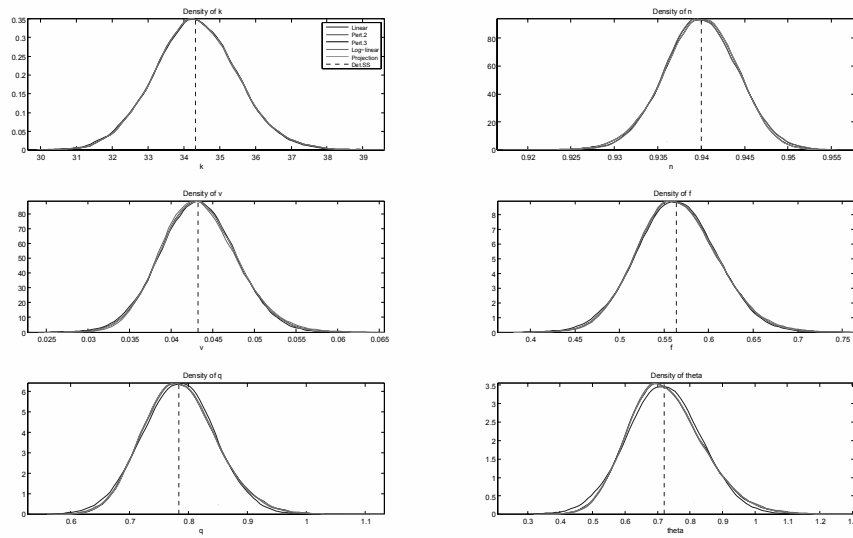
2.4.2 Simulated Density

Before performing accuracy tests, all the approximations are simulated for the estimation of density. Such simulated density indicates, as noted in Aruoba et al. (2006), a plausible range of the state space in which accuracy test like the Euler equation error test is conducted. For local approximations like the perturbations, such indicated ranges of the state space are particularly useful in evaluating their ability of producing global implications.

Each approximation is simulated once, with 101,000 observations and the first 1000 discarded. For comparison, all approximations are fed with the same sequence of exogenous shocks in simulation with which the density is estimated based on a normal kernel function.

Figure 2.1 depicts the simulated density of the two endogenous state variables, i.e., capital and employment, and other labor market variables. Note that for each variable, the simulated densities based on different approximations are similar and roughly centered around the deterministic steady state. Capital and employment range from 29 to 40 and from 0.90 to 0.96 respectively. The Euler equation error test will accordingly be conducted on such ranges. Besides, the simulated density of vacancy filling rate q shows that under the calibration in section 2.3.3, most of the realizations of this variable fall in between 0.6 and 1, exceeding unity very infrequently. Moreover, given the Cobb-Douglas matching function and the values of η and m_0 , the realizations of q that are smaller than m_0 correspond to those of labor market tightness θ that are larger than one. This implies that this calibrated model allows the vacancies to outnumber the unemployment workers, whereas it still captures the uncoordinated nature of the search process as the job finding rate f does not exceed unity as shown by its simulated

Fig. 2.1 Simulated Density



density.¹⁴

Petrosky-Nadeau and Zhang (2013) have noted that, when solving Hagedorn and Manovskii's (2008) model using den Haan and Marcet's (1990) parameterized expectations algorithm with a spline basis, the vacancy rate can fall below zero at nevertheless an extremely low frequency, and therefore incorporated a nonnegativity constraint on vacancy in their characterization of the model. Albeit the labor market in the model economy resembles that described by Hagedorn and Manovskii (2008) in many respects, the simulated density of v shows that the realization of vacancy remains positive at all frequencies, centering at its deterministic steady state value 0.043 and ranging from about 0.02 to 0.07, which covers roughly 50% deviation from the steady state on each side. Given that the model generates about 1% deviation in labor productivity from its steady state, this range of vacancy is sufficiently large to accommodate the empirical observation that the vacancy is about 10 times more volatile than the labor productivity as reported by Shimer (2005).

¹⁴Andofatto (1996) formulates this uncoordinated nature of the search process as $M(v, (1-n)) \leq \min\{v, (1-n)\}$, which implies $M(v, (1-n))/(1-n) \equiv f(\theta) \leq \min\{\theta, 1\}$ with the constant return to scale assumption on the matching function $M(v, (1-n))$. Therefore, when $\theta > 1$, the search friction still exists and is nontrivial if $f(\theta) < 1$.

2.4.3 Den Haan and Marcet's (1994) Accuracy Test

All the approximated solutions are firstly sent to Den Haan and Marcet's (1994) accuracy test to evaluate their performance in a dynamic and simulation-based environment. To examine how well the approximations satisfy the Euler equation for consumption and employment respectively, the test statistics are calculated and reported separately for the two Euler equations. Starting with the consumption Euler equation, inserting the functional form of the marginal consumption (5.20) and capital productivity (5.25) in (5.23) yields

$$c_t^{-1} = \mathbb{E}_t [\beta c_{t+1}^{-1} (\alpha e^{\tilde{z}_{t+1}} k_{t+1}^{\alpha-1} n_{t+1}^{\alpha-1} + 1 - \delta)] \quad (2.34)$$

Defining the expression in the expectation operator as a new variable

$$\phi_{t+1} \equiv \beta c_{t+1}^{-1} (\alpha e^{\tilde{z}_{t+1}} k_{t+1}^{\alpha-1} n_{t+1}^{\alpha-1} + 1 - \delta) \quad (2.35)$$

Then the forecast error of ϕ_{t+1} writes

$$u_{t+1} = \mathbb{E}_t(\phi_{t+1}) - \phi_{t+1} = c_t^{-1} - \phi_{t+1} \quad (2.36)$$

If the solution were exact, then u_{t+1} would have zero mean, and satisfy the following

$$\mathbb{E}[u_{t+1} \otimes h(x_t)] = 0 \quad (2.37)$$

for any function $h: \mathbb{R}^k \rightarrow \mathbb{R}^q$ and for any k -dimensional vector x_t belongs to the information set on which the conditional expectation in the Euler equation (2.34) is formed. To evaluate the performance of an approximation, inserting its simulation in the sample analog of the previous equation

$$M_T = (1/T) \sum_{t=1}^T u_{t+1}^{sim} \otimes h(x_t^{sim}) \quad (2.38)$$

where sim indicates the corresponding simulated series and T the length of simulation, and checking if M_T is close to zero. Note that, M_T could be made small by taking a $h(\cdot)$ with small function values, and owing to sampling error, M_T will not be exactly equal to zero. To avoid such problems, Den Haan and Marcet (1994) construct the following test statistic, with the null hypothesis that the approximation under evaluation is accurate, i.e., (2.37) holds for this approximation, to examine if M_T is significantly different from zero

$$J_T = T M_T' W_T^{-1} M_T \quad (2.39)$$

where W_T is some weighting matrix, chosen to take the following form

$$W_T = (1/T) \sum_{t=1}^T \left[(u_{t+1}^{sim} \otimes h(x_t^{sim})) (u_{t+1}^{sim} \otimes h(x_t^{sim}))' \right] \quad (2.40)$$

When the solution is exact and T goes to infinity, J_T converges to a χ^2 distribution with, as the Euler equation (2.34) is of dimension 1×1 , $q \times 1$ degrees of freedom. If

the value of J_T of an approximation falls in the lower or upper critical region of the χ^2 distribution, then there is evidence against the accuracy of that approximation. The test statistic for the employment Euler equation can be constructed following the steps above¹⁵: inserting the functional form of the marginal disutility of labor (5.27), labor productivity (5.26) and two first derivatives of the matching function (5.28) and (5.29) in (5.24) and noting the definition of q_t , f_t and θ_t yields

$$\frac{\kappa_v}{(1-\eta)q_t c_t} = \mathbb{E}_t \left[\frac{\beta}{c_{t+1}} \left(-\frac{c_{t+1}}{n_{t+1}^{1/\gamma}} + (1-\alpha)e^{\tilde{z}_{t+1}} \left(\frac{k_{t+1}}{n_{t+1}} \right)^\alpha + \kappa_u + \frac{\kappa_v(1-\chi-\eta f_{t+1})}{(1-\eta)q_{t+1}} \right) \right] \quad (2.41)$$

Defining the expression in the expectation operator as

$$\phi_{t+1} \equiv \frac{\beta}{c_{t+1}} \left(-\frac{c_{t+1}}{n_{t+1}^{1/\gamma}} + (1-\alpha)e^{\tilde{z}_{t+1}} \left(\frac{k_{t+1}}{n_{t+1}} \right)^\alpha + \kappa_u + \frac{\kappa_v(1-\chi-\eta f_{t+1})}{(1-\eta)q_{t+1}} \right) \quad (2.42)$$

and the forecast error of ϕ_{t+1} writes

$$u_{t+1} = \frac{\kappa_v}{(1-\eta)q_t c_t} - \phi_{t+1} \quad (2.43)$$

Inserting the involved simulated series in the previous equation yields u_{t+1}^{sim} with which the test statistic as given in (2.39) can be constructed for the employment Euler equation.

As noted by Aruoba et al. (2006), the null hypothesis will be rejected for all approximations if T is sufficiently large. On the other hand, Den Haan and Marcet (1994) note that an accurate/inaccurate approximation could fail/pass the test with a plausible T simply by chance. To control for such problems, each approximation is simulated 1000 times and each simulation contains 1000 observations with first 500 discarded. These 1000 simulations produce 1000 J_T values for each approximation and the percentages of the J_T values in the upper and lower 5% critical regions of the distribution are documented. For an accurate approximation, both the two percentages should be close to 5 as noted by Aruoba et al. (2006). An approximation is considered inaccurate, however, if its J_T value falls in the upper 5% region too often, and rarely drops in the lower 5% region.

Table 2.3 reports the test results. As can be seen, all the approximations satisfy the consumption Euler equation well, since all the percentages in column 2 and 3 of the table are close to 5. Meanwhile, as all the percentages in these two columns are similar in value, it is so far unclear which solution method is preferred in terms of accuracy. For the employment Euler equation, however, projection provides the most accurate approximation, outperforming perturbation of all three orders, either in levels

¹⁵To save notation, ϕ_{t+1} and u_{t+1} are recycled from (2.35) and (2.36), and will be redefined below.

Table 2.3 DHM Accuracy Test, $T = 500$

	J_T for Consumption Euler		J_T for Employment Euler	
	< 5%	> 95%	< 5%	> 95%
Linear	4.5	6.2	4.2	12.0
Log-linear	5.6	6.2	2.0	40.8
Perturbation 2	4.5	5.5	4.5	9.5
Perturbation 2 in Log	6.4	6.0	5.7	11.0
Perturbation 3	4.5	5.6	4.6	10.1
Perturbation 3 in Log	6.4	6.0	6.5	11.0
Projection	5.4	5.2	5.4	5.3

or in logs, as indicated by the percentages in the last two columns.

Among all the perturbation approximations for the employment Euler equation, the first order perturbation in logs (log-linearization) is the least accurate one since its J_T falls in the upper critical region too often (40.8 percent) and seldomly drops in the lower critical region (2.0 percent). Still at first order, the approximation in levels (linearization) achieves a much higher degree of accuracy with the upper tail percentage down to 12 and lower tail percentage rising to 4.2. Aruoba et al. (2006) have also observed, when they compare solution methods for a real business cycle model with endogenous labor choice, that linear approximation outperforms log-linearization, contradicting to the common practice. In comparison with linear approximations, second and third order perturbation further drives down the upper tail percentage, exhibiting a higher degree of accuracy.

Den Haan and Marcet's (1994) test evaluates how well the simulation of an approximation fits the Euler equations, and therefore has an implication for the accuracy of the simulation-based results like simulated moments. Moreover, the construction of the test statistic requires no approximation of the conditional expectation, which could be a potential source of inaccuracy in addition to that in the approximation itself.¹⁶ One drawback of the test is that there is no economic interpretation of the test result. The Euler equation error test in the next section presents the results that are economically interpretable.

¹⁶As noted by Judd (1992), the conditional expectation in the Euler equation involves an integral that cannot in general be evaluated explicitly and usually approximated with a finite sum.

2.4.4 Euler Equation Error Test

The Euler equation error test from Judd (1992) and Judd (1998) examines if the policy function is consistently approximated over two consecutive periods by evaluating a unit-free measure that expresses the one-period optimization error in relation to current consumption. Given the recursive structure of the Euler equation, current consumption can be written as a function of the next period consumption and other model variables: for the consumption Euler equation, rearranging (2.34) yields

$$c_t = (\mathbb{E}_t [\beta c_{t+1}^{-1} (\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} n_{t+1}^{\alpha-1} + 1 - \delta)])^{-1} \quad (2.44)$$

Likewise, for the employment Euler equation, rearranging (2.41) yields

$$c_t = \left(\mathbb{E}_t \left(\frac{\beta(1-\eta)q_t}{\kappa_v c_{t+1}} \left(-\frac{c_{t+1}}{n_{t+1}^{1/\gamma}} + (1-\alpha)e^{z_{t+1}} \left(\frac{k_{t+1}}{n_{t+1}} \right)^\alpha + \kappa_u + \frac{\kappa_v(1-\chi-\eta f_{t+1})}{(1-\eta)q_{t+1}} \right) \right) \right)^{-1} \quad (2.45)$$

Inserting the involved approximations in the right hand side of the previous two equations yields the current consumption implied by the approximated, next period consumption and other approximated model variables

$$\hat{c}_t^{implied, ConEuler} = (\hat{\mathbb{E}}_t [\beta \hat{c}_{t+1}^{-1} (\alpha e^{\hat{z}_{t+1}} \hat{k}_{t+1}^{\alpha-1} \hat{n}_{t+1}^{\alpha-1} + 1 - \delta)])^{-1} \quad (2.46)$$

$$\hat{c}_t^{implied, EmpEuler} = \left(\hat{\mathbb{E}}_t \left(\frac{\beta(1-\eta)\hat{q}_t}{\kappa_v \hat{c}_{t+1}} \left(-\frac{\hat{c}_{t+1}}{\hat{n}_{t+1}^{1/\gamma}} + (1-\alpha)e^{\hat{z}_{t+1}} \left(\frac{\hat{k}_{t+1}}{\hat{n}_{t+1}} \right)^\alpha + \kappa_u + \frac{\kappa_v(1-\chi-\eta \hat{f}_{t+1})}{(1-\eta)\hat{q}_{t+1}} \right) \right) \right)^{-1} \quad (2.47)$$

where $\hat{\cdot}$ over the conditional expectation indicates this expectation has been explicitly approximated, as in Judd (1992), using the Gauss-Hermite quadrature method with the same number of quadrature points as used in the projection method discussed in section 2.3.2 to compute the coefficients Θ_c and Θ_v . The superscripts $ConEuler$ and $EmpEuler$ indicate the two implied current consumption are computed using the relationship given by the consumption and employment Euler equation, (2.44) and (2.45), respectively.

The test statistic is essentially the difference between the implied and the actual approximated current consumption, normalized as the common logarithm of the absolute value of the difference between unity and the ratio of the implied to the actual approximated current consumption

$$EEE^{ConEuler} = \log_{10} \left| 1 - \frac{\hat{c}_t^{implied, ConEuler}}{\hat{c}_t} \right| \quad (2.48)$$

$$EEE^{EmpEuler} = \log_{10} \left| 1 - \frac{\hat{c}_t^{implied, EmpEuler}}{\hat{c}_t} \right| \quad (2.49)$$

The two statistics above are computed at each and every point on a grid of the three

state variables, i.e., capital, employment and productivity. This test grid shares the same upper and lower bounds with the grid used by the projection method in section 2.3.2. However, it contains simply equispaced points (100 for capital, 100 for employment and 80 for productivity) that are not necessarily the collocation points. In other words, for the projection approximation, its accuracy is evaluated at the set of points other than the set on which the policy function is approximated. The two sets may nevertheless partially overlapping. Deviations in (2.48) and (2.49) from zero are interpreted by Judd (1992) and many others as the relative optimization error that results from using a particular approximation. $EEE = -1$ implies a one dollar error for every ten dollars spent and $EEE = -3$ implies a one dollar error for every thousand dollars spent.

Figure 2.2 depicts the consumption and the employment Euler equation error (the upper and the lower panel of the figure respectively) of the projection approximation in the capital-employment space. In this and all the other figures throughout the rest of this section, productivity is held at its steady state value (zero) unless otherwise specified. Since the policy function is approximated at the chosen collocation points, higher accuracy is achieved at and in the vicinity of those points: in the figure there is a lattice of high accuracy. The points where the edges of the lattice meet are the collocation points. Aside from this lattice, the projection approximation demonstrates a high degree of accuracy around the deterministic steady state. The quality of the approximation decreases, as capital and employment move away from their respective steady state value. In the area where capital and employment are both very high/low, the approximation reaches its lowest accuracy level.

Since the consumption and the employment Euler equation error are both expressed in relation to the same approximated current consumption, $EEE^{ConEuler}$ and $EEE^{EmpEuler}$ as given by (2.48) and (2.49) are directly comparable. Figure 2.3 depicts the difference between the consumption and the employment Euler equation error, i.e., $EEE^{ConEuler} - EEE^{EmpEuler}$ of the projection approximation. But for a few points the difference is smaller than zero in the entire capital-employment space. This implies that, with the projection approximation, the consumption Euler equation is in general better satisfied than the employment Euler equation.

Figure 2.4 plots the consumption and the employment Euler equation error of the third order perturbation in levels. This (and all the other perturbation) approximation is built around the deterministic steady state. As capital and employment deviate from their respective steady state value, the quality of approximation deteriorates. Like the projection approximation, the third order perturbation satisfies the consumption Euler

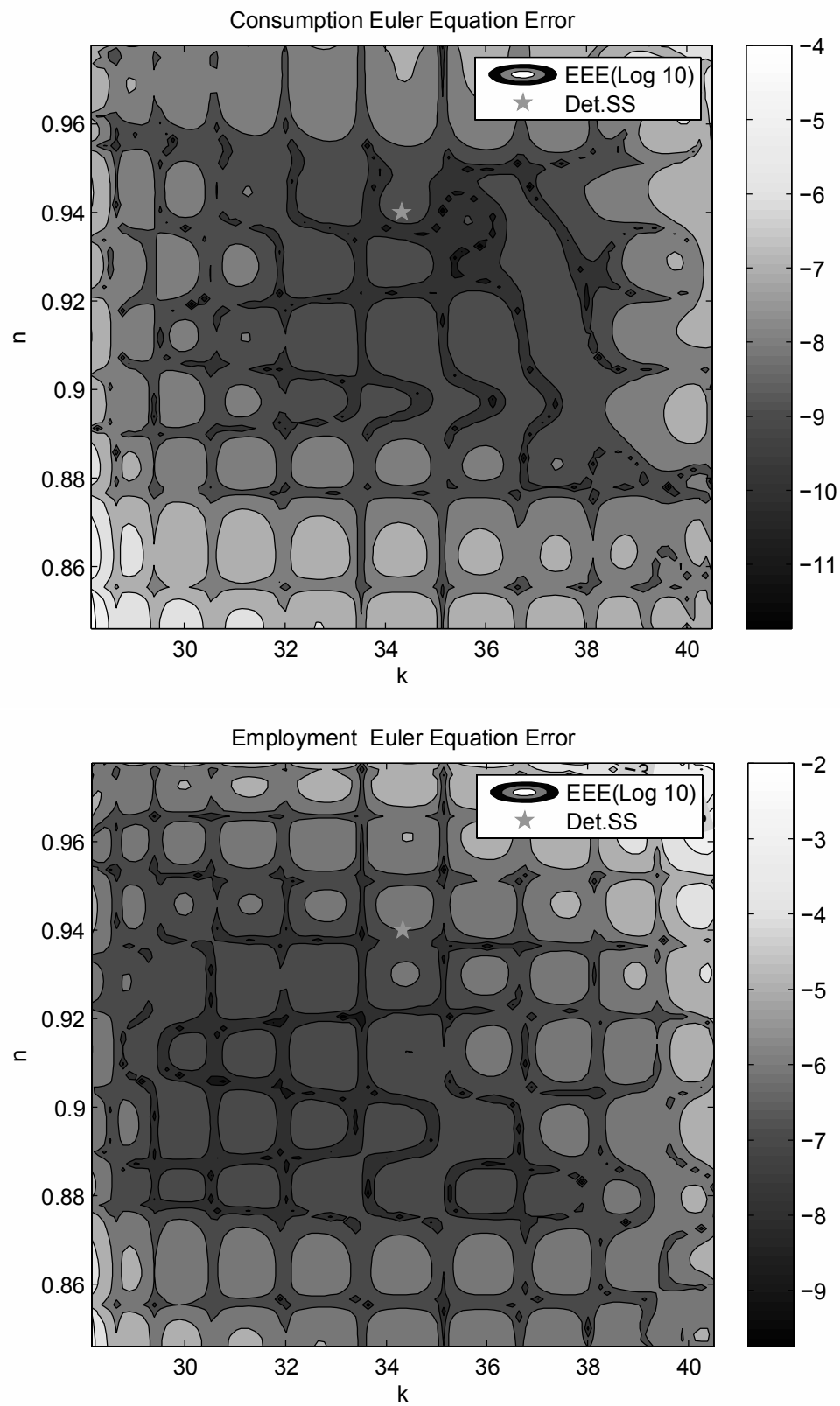
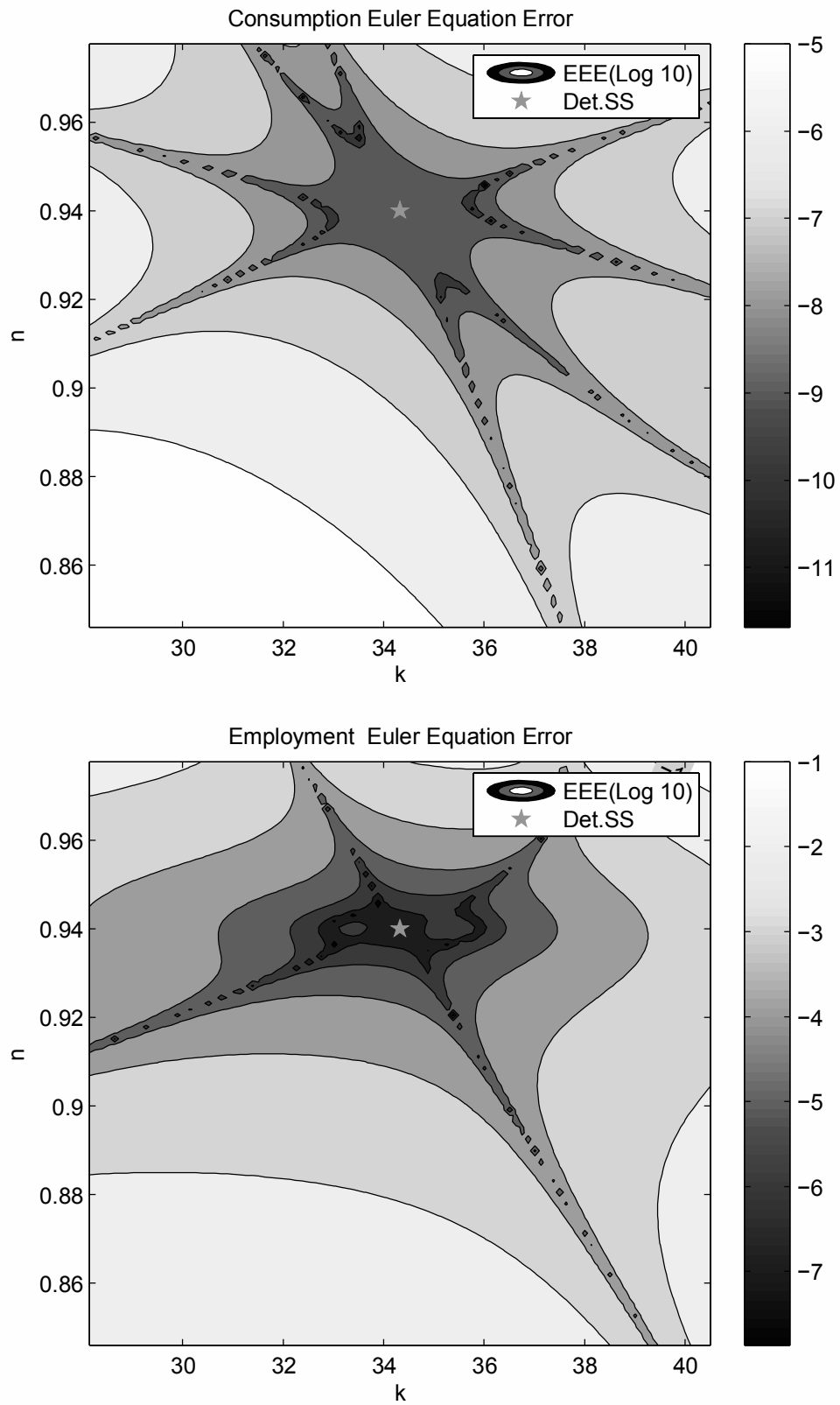
Fig. 2.2 EEE of Projection, $z = 0$ 

Fig. 2.3 Difference in EEE, $z = 0$ 

Fig. 2.4 EEE of Projection, $z = 0$ 

equation better than the employment Euler equation, as the difference, $EEE^{ConEuler} - EEE^{EmpEuler}$, is negative everywhere in the capital-employment space, see figure 2.5 below

Fig. 2.5 Difference in EEE, $z = 0$



To evaluate all the approximations and compare their performance on the entire three dimensional grid, the maximum and average Euler equation error are computed as in Judd (1992) and many others. Table 2.4 reports the results

There are three important observations. First, all the approximations satisfy the consumption Euler equation better than the employment Euler equation, measured by both the max and the average error. Second, the projection approximation performs better than all the perturbations in terms of the average error. This is not surprising, as all the perturbations are local approximations, built around only one point, i.e., the deterministic steady state on the grid. The projection method, however, allows its approximation to anchor on as many points (the collocation points) as desired on the grid, and therefore has a better global performance. Third, among all the perturbations and for the consumption Euler equation, higher order (for both level and log specifications)

Table 2.4 Euler Equation Error Test

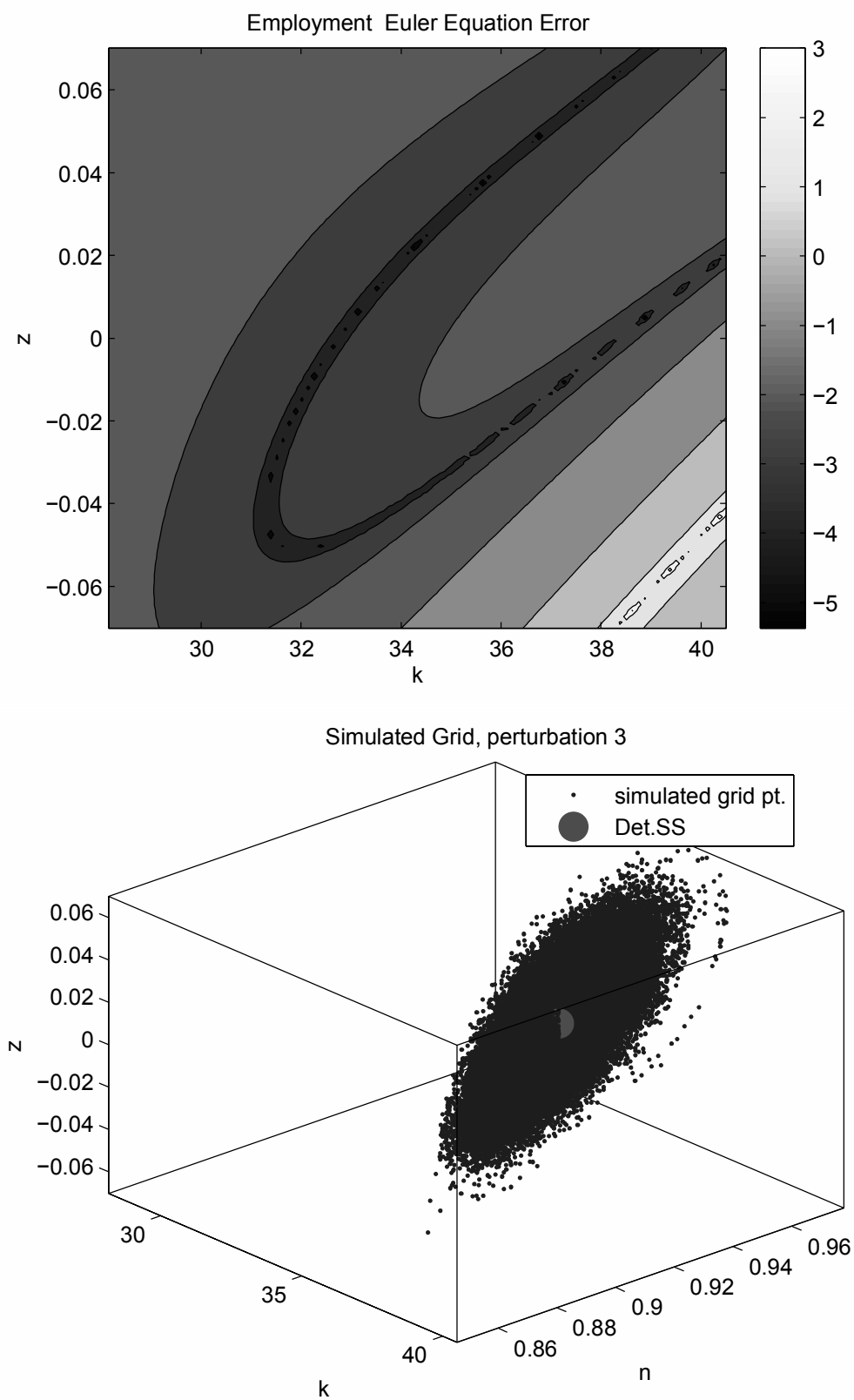
	Consumption Euler		Employment Euler	
	max. error	avg. error	max. error	avg. error
Linear	-3.25	-3.98	2.22	-0.47
Log-linear	-3.07	-3.91	0.45	-0.64
Perturbation 2	-3.63	-4.86	4.17	-0.93
Perturbation 2 in Log	-4.11	-4.98	4.24	-0.78
Perturbation 3	-3.93	-5.27	3.30	-1.14
Perturbation 3 in Log	-4.12	-5.25	3.25	-0.68
Projection	-2.95	-5.50	3.70	-1.79

performs uniformly better than the preceding order. Between level and log specification, the first order approximation in levels is superior to the first order approximation in logs, in line with Aruoba et al. (2006). Yet this relationship is reversed at the second order and moving to the third order, the approximation in levels again outperforms the approximation in logs but only on average.

Turning to the employment Euler equation, only the projection approximation and the third order perturbation in level are on average accurate. Yet the positive max errors suggest that none of the approximations is acceptable in some areas on the grid — at the grid point where the Euler equation error is positive, the ratio of the implied to the actual current consumption is negative, meaning there is no consistent consumption plan can be made over two consecutive periods.¹⁷ It is then important to know in which areas the employment Euler equation error goes above zero since some areas, as noted by Judd et al. (2010) and Judd et al. (2012), will never be visited in the equilibrium of the model. The Euler equation error computed in such areas, regardless of its sign and magnitude, contributes least to the evaluation of an approximation.

Using the third order perturbation as an example, the upper panel of figure 2.6 locates such areas on the grid by plotting the employment Euler equation error in the capital-productivity space and holding employment at its upper bound. In the neighborhood of the lower-right corner of the plot where the productivity lower bound meets the capital upper bound, given employment is at its upper bound, the error goes beyond 0 and up to 3. Note that, to push the productivity down to its lower bound requires a sequence of negative productivity shocks. Since simulated correlation based on the approximation suggests that both capital and employment are positively correlated with

¹⁷This is a qualitative inconsistency. To this end, a consistency is quantitative in nature if the corresponding Euler equation error is negative.

Fig. 2.6 Employment EEE of Perturbation 3 ($n = n_{\max}$) and Simulated Grid

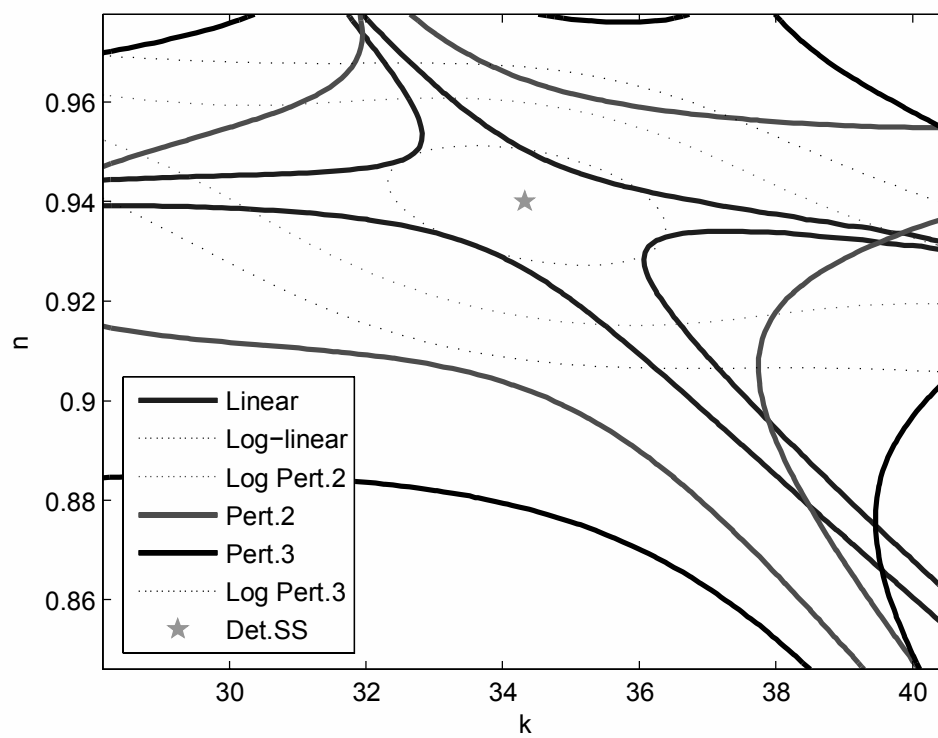
the shock, these two state variables would deviate from the deterministic steady state and move toward their respective lower, instead of upper bounds in response to such a sequence of shock realizations. As the lower panel of figure 2.6 shows, in simulation the model never hits the lower-right corner of the grid where $z = -0.06$ (its lower bound), $k = 42$ and $n = 0.98$ (the two upper bounds).¹⁸ The Euler equation error computed in this area appears therefore, not informative and even misleading as it increases the average error.

In this regard, Den Haan and Marcet's (1994) test presented in section 2.4.3 complements the Euler equation error test in evaluating the quality of the approximations of this model. Building its test statistic on the simulated time series in which the correlation among the state variables implied by the approximation has been taken into account, Den Haan and Marcet's (1994) test implicitly narrows down the test grid to the realized state space associated with the approximation. As table 2.3 reports, when examined using Den Haan and Marcet's (1994) test, both the projection approximation and the third order perturbation in level are accurate whereas the former is superior to the latter in the upper tail of the distribution.

Ignoring those redundant areas on the grid, the third order in levels outperforms all the other perturbations in satisfying the employment Euler equation. For comparison, figure 2.7 plots the -2 contours of the employment Euler equation error in the capital-employment space. For each perturbation, the area circled inside its -2 contour is the region where the employment Euler equation error is smaller than -2 . In terms of the size of this -2 accuracy area, the third order in levels dominates all the others. Moreover, for both level and log specifications, higher order in general performs better than the preceding order and at first order, linear approximation is -2 accurate on a larger area than that of log-linearization, which potentially contributes to understanding the result from Den Haan and Marcet's (1994) test at this order.

To summarize, the projection provides the most accurate approximation according to the Euler equation error test. All the approximations satisfy the consumption Euler equation better than the employment Euler equation. In addition, among all the perturbations, the third order in levels is the most accurate one, comparable to the projection approximation.

¹⁸To produce the simulated grid, all the approximations are simulated in the environment described in section 2.4.2.

Fig. 2.7 Employment EEE of Perturbations, -2 Contour

2.4.5 Simulated Moments Comparison

This section presents the moments computed using the simulated series based on different approximations. All the approximations are simulated in the same environment as that described in section 4.6.4. For all the level approximations (the projection and the perturbation in levels at all three order), their simulated series are transformed by the natural logarithm before applying the Hodrick-Prescott filter.

Table 2.5 Relative Standard Deviation from Data and Model

Statistic	Data	Model I	Method	Statistic	Data	Model I	Method
σ_c/σ_y	0.40	0.34	(PJ)	σ_u/σ_y	6.11	3.37	(PJ)
		0.34	(P3)			3.39	(P3)
		0.34	(P2)			3.40	(P2)
		0.34	(LN)			3.42	(LN)
		0.34	(LLN)			3.38	(LLN)
σ_k/σ_y	0.22	0.29	(PJ)	σ_v/σ_y	7.31	7.31	(PJ)
		0.29	(P3)			7.31	(P3)
		0.29	(P2)			7.34	(P2)
		0.29	(LN)			7.40	(LN)
		0.29	(LLN)			7.30	(LLN)
σ_n/σ_y	0.54	0.22	(PJ)	σ_θ/σ_p	19.10	10.32	(PJ)
		0.22	(P3)			10.32	(P3)
		0.22	(P2)			10.25	(P2)
		0.22	(LN)			10.72	(LN)
		0.22	(LLN)			10.31	(LLN)
σ_y	1.87	1.05	(PJ)	σ_p/σ_y	0.68	0.84	(PJ)
		1.05	(P3)			0.84	(P3)
		1.05	(P2)			0.84	(P2)
		1.05	(LN)			0.84	(LN)
		1.05	(LLN)			0.84	(LLN)

* PJ:projection, P3: 3rd order perturbation, P2: 2nd order perturbation, LN: linearization, LLN: log-linearization

Table 2.5 reports the standard deviation of the selected model variables relative to that of output or labor productivity. Taking those generated by simulating the projection approximation as the benchmark since the projection approximation outperforms all the perturbations in terms of accuracy, all the relative volatilities generated by perturbations are very close to the benchmark, and to each other. The volatility of

consumption, capital, employment and labor productivity in relation to that of output are even identical across all the approximations. The linear approximation tends to slightly overstate the relative volatility of vacancy and labor market tightness. For log-linearization, though it appears the least accurate approximation in terms of satisfying the employment Euler equation, the relative volatilities it generate are still very close to the benchmark.

Table 2.6 Correlation and Autocorrelation from Data and Model

Statistic	Data	Model I	Method	Statistic	Data	Model I	Method
$\rho(u, v)$	-0.8949	-0.1957	(PJ)	$\rho(u, \theta)$	-0.971	-0.554	(PJ)
		-0.2000	(P3)			-0.553	(P3)
		-0.1975	(P2)			-0.550	(P2)
		-0.1952	(LN)			-0.544	(LN)
		-0.1970	(LLN)			-0.556	(LLN)
$\rho(u, p)$	-0.408	-0.677	(PJ)	$\rho(v, \theta)$	0.975	0.925	(PJ)
		-0.677	(P3)			0.924	(P3)
		-0.676	(P2)			0.927	(P2)
		-0.674	(LN)			0.922	(LN)
		-0.677	(LLN)			0.924	(LLN)
$\rho(v, p)$	0.364	0.813	(PJ)	$\rho(\theta, p)$	0.396	0.953	(PJ)
		0.813	(P3)			0.953	(P3)
		0.814	(P2)			0.953	(P2)
		0.812	(LN)			0.946	(LN)
		0.815	(LLN)			0.955	(LLN)
$\rho(u_t, u_{t-1})$	0.936	0.795	(PJ)	$\rho(v_t, v_{t-1})$	0.940	0.329	(PJ)
		0.795	(P3)			0.329	(P3)
		0.795	(P2)			0.331	(P2)
		0.796	(LN)			0.328	(LN)
		0.796	(LLN)			0.329	(LLN)
$\rho(\theta_t, \theta_{t-1})$	0.941	0.597	(PJ)	$\rho(p_t, p_{t-1})$	0.878	0.660	(PJ)
		0.600	(P3)			0.660	(P3)
		0.595	(P2)			0.660	(P2)
		0.597	(LN)			0.660	(LN)
		0.599	(LLN)			0.660	(LLN)

* PJ:projection, P3: 3rd order perturbation, P2: 2nd order perturbation, LN: linearization, LLN: log-linearization

Moving to the (auto)correlation, as table 2.6 shows, the results from all the approxi-

mations are also very similar. This similarity among the simulated moments originates from the similarity among all the approximations in the neighborhood of the deterministic steady state and most frequently, the realizations of the model fall in that region.

Figure 2.8 plots, for example, the approximated policy function of the vacancy and the labor market tightness from the (log)linear approximation, the third order perturbation in levels and the projection on the employment grid, holding the other two state variables (capital and productivity) at their respective steady state value. In addition, the histogram of employment has been appended to the plot in order to show the distribution of the employment realizations.¹⁹ The approximated policy function implies, in the vicinity of the steady state employment, that is, between 0.92 and 0.96, the corresponding values of the vacancy and the labor market tightness indicated by the four approximations are very similar, and this vicinity, as the histogram shows, happens to be the region in which most of the employment realizations fall. The simulated series and therefore the simulated moments, are accordingly similar across the four approximations.

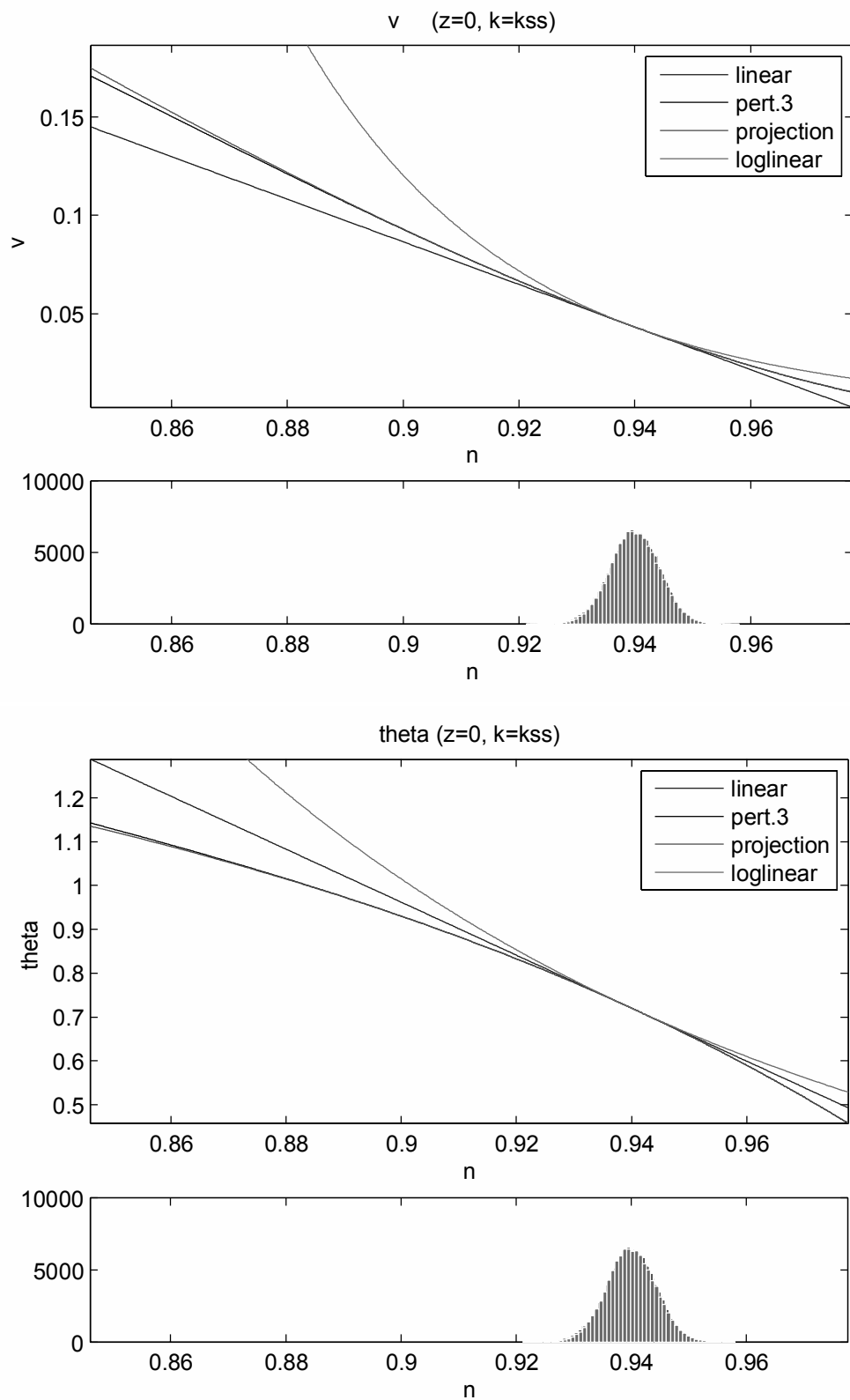
2.5 Conclusion

In this paper I have solved a real business cycle model with labor market search frictions using the projection and the perturbation methods under the conventional quarterly calibration. I then implement Den Haan and Marcet's (1994) test and the Euler equation error test from Judd (1992) and Judd (1998) to evaluate the quality of all the approximated solutions. The results from the two tests suggest that the approximation provided by the projection method is the most accurate among all the approximations, and the third order perturbation in levels also achieves a degree of accuracy comparable to that of the projection approximation. Among all the perturbations and for both log and level specifications, the results from the Euler equation error test show that, higher order performs on average better than the preceding order.

By comparing the respective test statistic for the consumption and the employment Euler equation, I find that across all the approximations, the consumption Euler equation is better satisfied than the employment Euler equation. Moreover, the results from Den Haan and Marcet's (1994) test suggest that the first order perturbation in levels is preferred to the first order perturbation in logs in satisfying the employment Euler equation. In satisfying the consumption Euler equation, the results from the Euler

¹⁹To produce the histogram, all the approximations are simulated in the environment described in section 2.4.2.

Fig. 2.8 Approx. Policy Rule and Histogram



equation error test also indicates that the level specification performs better than the log specification at first order.

To analyze the implications of the difference in accuracy among all the approximations, I compare the simulated moments based on different approximations and find that all of them are similar in value. Even for the approximations with a relatively low degree of accuracy such as the first order perturbation in levels and in logs, the simulated moments produced by them are very close to those produced by the projection approximation. To explain this similarity, I simulate all the approximations and present the resulting histogram of their realizations and find that, for all the approximations, most of their realizations fall in the neighborhood of the deterministic steady state of the model and in this neighborhood, all the approximations behave similarly.

2.6 Acknowledgements

I am grateful to Michael Burda, Lutz Weinke, Alexander Meyer-Gohde and Julien Albertini as well as participants of research seminars at HU Berlin for discussions. This research was supported by the DFG through the SFB 649 “Economic Risk”. Any and all errors are entirely my own.

Chapter 3

Solving DSGE Models with a Nonlinear Moving Average

Hong Lan[†]

Alexander Meyer-Gohde[§]

Abstract

We propose a nonlinear infinite moving average as an alternative to the standard state space policy function for solving nonlinear DSGE models. Perturbation of the nonlinear moving average policy function provides a direct mapping from a history of innovations to endogenous variables, decomposes the contributions from individual orders of uncertainty and nonlinearity, and enables familiar impulse response analysis in nonlinear settings. When the linear approximation is saddle stable and free of unit roots, higher order terms are likewise saddle stable and first order corrections for uncertainty are zero. We derive the third order approximation explicitly, examine the accuracy of the method using Euler equation tests, and compare with state space approximations.

JEL classification: C61, C63, E17

Keywords: Perturbation; nonlinear impulse response; DSGE; solution methods; Volterra series

[†]Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 1466; Email: lanhong@cms.hu-berlin.de

[§]Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 5720; Fax: +49-30-2093 5696; E-Mail: alexander.meyer-gohde@wiwi.hu-berlin.de

3.1 Introduction

Solving models with a higher than first order degree of accuracy is an important challenge for DSGE analysis with the growing interest in nonlinearities. We introduce a novel policy function, the nonlinear infinite moving average, to perturbation analysis in dynamic macroeconomics. This direct mapping from shocks to endogenous variables neatly dissects the individual contributions of orders of nonlinearity and uncertainty to the impulse response functions (IRFs). For economists interested in studying the transmission of shocks, our method offers new insight into the propagation mechanism of nonlinear DSGE models.

The nonlinear moving average policy function chooses as its state variable basis the infinite history of past shocks.¹ The nonlinear DSGE perturbation literature initiated by Gaspar and Judd (1997), Judd and Guu (1997), and Judd (1998, ch. 13) has thus far operated solely with state space methods.² Our infinite dimensional approach is longstanding in linear models and delivers the same solution as state space methods for linear models.³ For the nonlinear focus of this paper, however, it provides a different solution. Deriving the direct mapping from shocks to endogenous variables—a Volterra series expansion—facilitates familiar impulse response analysis and makes clear the caveats introduced by nonlinearity. These include history dependence, asymmetries, a breakdown of superposition and scale invariance, as well as harmonic distortion.⁴

As highlighted by Gomme and Klein (2011) in their second order approximation, deriving perturbation solutions with standard linear algebra increases the transparency of the technique and makes coding the method more straightforward. In that vein, we adapt Vetter’s (1973) multidimensional calculus to provide a mechanical system of differentiation that maintains standard linear algebraic structures for arbitrarily high orders of approximation. We implement our approach numerically by providing an add-on for the popular Dynare package.⁵ We then apply our method to the stochas-

¹Kalman’s (1980) “external” or “empirical” approach to system theory in contrast to the ‘internal’ or “state-variable” approach of the state space methods currently more familiar to DSGE practitioners. See Woodford (1986) for a theoretical foundation of nonlinear DSGE solutions in this space of infinite sequences of innovations.

²See Collard and Juillard (2001a), Collard and Juillard (2001b), Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), Anderson et al. (2006), Lombardo and Sutherland (2007), and Kim et al. (2008). Recent work of Aruoba et al. (2012) links their quadratic autoregressive (QAR) time series model within a DSGE context to the Volterra series expansion that we use as our solution basis.

³Compare, e.g., the state space representations of Uhlig (1999), Klein (2000), or Sims (2001) with the infinite moving-average representations of Muth (1961), Whiteman (1983) or Taylor (1986).

⁴See also Priestly (1988), Koop et al. (1996), Potter (2000), and Gourieroux and Jasiak (2005).

⁵See Adjemian et al. (2011) for Dynare. Our add-on can be downloaded at <http://www.wiwi>.

tic growth model of Aruoba et al. (2006) for comparability and explore the resulting decomposition of the contributing components of the responses of variables to exogenous shocks. We develop Euler equation error methods for our infinite dimensional policy function and confirm that our moving average solution produces approximations with a degree of accuracy comparable to state space solutions of the same order of approximation presented in Aruoba et al. (2006).⁶

We make two assumptions on the first order (i.e., linear) approximation: it is saddle stable and it is free of unit roots. The first is the standard Blanchard and Kahn (1980) assumption and we show that the resulting stability from the first order is passed on to higher order terms. The second ensures the boundedness of corrections to constants and the two together guarantee the local invertibility of a standard state space policy function to yield our infinite moving average.

The paper is organized as follows. The model and the nonlinear infinite moving average policy function are presented in section 3.2. In section 3.3, we develop the numerical perturbation of our nonlinear infinite moving average policy function explicitly out to the third order. We compare our policy function with state space policy functions in section 3.4. We apply our method to a standard stochastic growth model in section 3.5, a widely used baseline for numerical methods in macroeconomics. In section 3.6, we develop Euler equation error methods for our infinite dimensional solution form and quantify the accuracy of our method. Section 3.7 concludes.

3.2 Problem Statement and Solution Form

We begin by introducing our class of models, a standard system of (nonlinear) second order expectational difference equations. In contrast with the general practice in the literature, however, the solution will be a policy function that directly maps from realizations of the exogenous innovations to the endogenous variables of interest. We then approximate the solution with a Volterra series and present the matrix calculus used in subsequent sections.

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⁶Aruoba et al. (2006) also explore several global methods (projection, value function iteration) and our choice allows comparability to these other methods. Our focus is on the alternative basis from the nonlinear moving average for local (perturbation) methods and we proceed accordingly.

3.2.1 Model Class

We analyze a family of discrete-time rational expectations models given by

$$0 = E_t[f(y_{t-1}, y_t, y_{t+1}, u_t)], \text{ where } u_t = \sum_{i=0}^{\infty} N^i \varepsilon_{t-i} \quad (3.1)$$

f is an $(neq \times 1)$ vector valued function, continuously M -times (the order of approximation to be introduced subsequently) differentiable in all its arguments; y_t is an $(ny \times 1)$ vector of endogenous variables; the vector of exogenous variables u_t is of dimension $(nu \times 1)$ and it is assumed that there are as many equations as endogenous variables ($neq = ny$). N is the $(nu \times nu)$ matrix of autoregressive coefficients of u_t , presented here in moving average form. The eigenvalues of N are assumed all inside the unit circle so that u_t admits this infinite moving average representation; and ε_t is an $(ne \times 1)$ vector of exogenous shocks of the same dimension ($nu = ne$).⁷ ε_t is assumed independently and identically distributed such that $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^{\otimes [n]})$ exists and is finite for all n up to and including the order of approximation to be introduced subsequently.⁸

As is usual in perturbation methods, we introduce an auxiliary parameter $\sigma \in [0, 1]$ to scale the uncertainty in the model. The value $\sigma = 1$ corresponds to the “true” stochastic model under study and $\sigma = 0$ represents the deterministic version of the model.⁹ Following Anderson et al. (2006, p. 4), we do not scale $\{\varepsilon_t, \varepsilon_{t-1}, \dots\}$ —the realizations of the exogenous shocks up to (including) t —with σ , as they are known with certainty at t . The perturbation parameter does not enter the problem statement explicitly, but only implicitly through the policy functions, and its role will become clear as we introduce the solution form and its approximation.

Fleming (1971) and Jin and Judd (2002) emphasize that the use of σ to transition from the deterministic to the stochastic model depends crucially on the two models

⁷Our software add-on forces $N = 0$ to align with Dynare Adjemian et al. (2011). Thus in practice, the economist using Dynare must incorporate any exogenous serial correlation by including u_t in the vector y_t . This choice is not made in the exposition here as the admissibility of serial correlation in the exogenous driving force brings our first order derivation in line with earlier moving average approaches for linear models, e.g., Taylor (1986).

⁸The notation $\varepsilon_t^{\otimes [n]}$ represents Kronecker powers, $\varepsilon_t^{\otimes [n]}$ is the n 'th fold Kronecker product of ε_t with itself: $\underbrace{\varepsilon_t \otimes \varepsilon_t \cdots \otimes \varepsilon_t}_{n \text{ times}}$. For simulations, of course, more specific decisions regarding the distribution of

the exogenous processes will have to be made. Kim et al. (2008, p. 3402) emphasize that distributional assumptions like these are not entirely local assumptions. Dynare Adjemian et al. (2011) assumes normality of the underlying shocks.

⁹Our formulation follows Adjemian et al.'s (2011) Dynare, Anderson et al.'s (2006) Perturbation-AIM and Juillard (2011). This is not the only way to perturb the model: Lombardo (2010), for example, scales the entire history of shocks $\{\varepsilon_t, \varepsilon_{t-1}, \dots\}$ along with the unrealized future shocks. See section 3.4 for further discussion.

being “close,” in the sense that the underlying uncertainty scaled by σ is “small,” as a stochastic perturbation like this is singular in that it changes the underlying order of the problem, see Judd (1998, ch. 13). Kim et al. (2008) note the importance of the “underlying assumption” of sufficient differentiability within a neighborhood of $\sigma = 0$ and Anderson et al. (2006) simply make the explicit assumption that the policy function, the solution to be introduced in the following subsection, is analytic within a domain that encompasses $\sigma = 0$ and $\sigma = 1$, enabling its representation in σ by a Taylor series evaluated anywhere within that domain. Deriving explicit conditions for the model with $\sigma = 1$ to be sufficiently close to the $\sigma = 0$ model is beyond the scope of our study here and we follow the literature by assuming that a local approach to σ remains valid as we transition to the stochastic model.

3.2.2 Solution Form

Let the policy function take the causal one-sided infinite sequence of shocks as its state vector and, following Anderson et al. (2006, p. 3), let it be time invariant for all t , analytic and ergodic.¹⁰ The unknown policy function is then given by

$$y_t = y(\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots) \quad (3.2)$$

Note that σ enters as a separate argument. As the scale of uncertainty changes, so too will the policy function y itself change. Time invariance and scaling uncertainty give us

$$y_{t-1} = y^-(\sigma, \varepsilon_{t-1}, \varepsilon_{t-2}, \dots) \quad (3.3)$$

$$y_{t+1} = y^+(\sigma, \tilde{\varepsilon}_{t+1}, \varepsilon_t, \varepsilon_{t-1}, \dots), \text{ where } \tilde{\varepsilon}_{t+1} \equiv \sigma \varepsilon_{t+1} \quad (3.4)$$

The notation, y , y^- , and y^+ , is adopted so that we can keep track of the source (through y_t , y_{t-1} , and y_{t+1} respectively) of any given partial derivative of the policy function. Due to the assumption of time invariance, y , y^- , and y^+ are the same function differing only in the timing of their arguments. The importance of discriminating among these functions will become clear in the next section. The term $\sigma \varepsilon_{t+1}$ in (3.4) is the source of uncertainty, via ε_{t+1} , that we are perturbing with σ . The known function u of the exogenous variable is written similarly as

$$u_t = u(\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots) = \sum_{i=0}^{\infty} N^i \varepsilon_{t-i} \quad (3.5)$$

For notational ease, we will define vector x_t , containing the complete set of vari-

¹⁰Analyticity guarantees the convergence of the asymptotic expansion to the true solution, see the discussion regarding σ above, within the domain of convergence as the order of approximation becomes infinite and ergodicity rules out explosive and nonfundamental solutions.

ables

$$x_t \equiv \begin{bmatrix} y'_{t-1} & y'_t & y'_{t+1} & u'_t \end{bmatrix}' \quad (3.6)$$

x_t is of dimension $(nx \times 1)$ with $(nx = 3ny + ne)$. With the policy function of the form (3.2), (3.3) and (3.4), plus the function of the exogenous variable (3.5), we can write x_t as

$$x_t = x(\sigma, \tilde{\varepsilon}_{t+1}, \varepsilon_t, \varepsilon_{t-1}, \dots) \quad (3.7)$$

Following from the assumptions on y and u , x is likewise time invariant, analytic and ergodic.

3.2.3 Approximation: Taylor/Volterra Series Approximation

We will approximate the solution, (3.2), as a Taylor series in the infinite state vector (i.e., a Volterra series) expanded around a deterministic steady state, \bar{x} , the solution to (4.22) with all shocks, past and present, set to zero and all uncertainty regarding the future eliminated ($\sigma = 0$)

Definition 3.2.1. *Deterministic Steady State*

Let $\bar{y} \in \mathbb{R}^{ny}$ be a vector such that

$$0 = f(\bar{y}, \bar{y}, \bar{y}, 0) = f(\bar{x}) \quad (3.8)$$

Furthermore, $\bar{y} = y(0, 0, \dots)$ is the solution (3.2) evaluated at the deterministic steady state.

Following general practice in the perturbation literature, we pin down the approximation of the unknown policy function (3.2) by successively differentiating (4.22) and solving the resulting systems for the unknown coefficients. The algorithm is detailed in section 3.3. Notice that, since f is a vector valued function, successive differentiation of f with respect to its arguments, which are vectors in general, will generate a hypercube of partial derivatives. We adapt the structure of matrix derivatives defined in Vetter (1973) to unfold the hypercube conformable to the Kronecker product—allowing us to avoid tensor notation and use standard linear algebra—¹¹collecting partial derivatives from successive differentiation of f in two dimensional matrices as follows

Definition 3.2.2. *Matrix Derivatives*

Let $A(B, C)$ be a matrix-valued function that maps the real-valued $r \times 1$ and $s \times 1$

¹¹Related approaches can be found in Chen and Zdrozny (2003), Gomme and Klein (2011), and Binning (2013). Using a different method, Lombardo and Sutherland (2007) also derive a second order solution without appealing to tensor notation. We choose the approach of Vetter (1973) to be consistent with the derivatives provided by Dynare—see also Juillard (2011) for details—and, consequentially, our software add-on for Dynare.

vectors B and C into an $p \times q$ real-valued matrix $A(B, C)$, derivatives of $A(B, C)$ are

$$A_B \equiv \mathcal{D}_{B^T} \{A\} \equiv \left[\frac{\partial}{\partial b_1} \quad \dots \quad \frac{\partial}{\partial b_r} \right] \otimes A \quad (3.9)$$

where b_i denotes i 'th row of vector B , T indicates transposition; n 'th derivatives are

$$A_{B^n} \equiv \mathcal{D}_{(B^T)^n} \{A\} \equiv \left(\left[\frac{\partial}{\partial b_1} \quad \dots \quad \frac{\partial}{\partial b_r} \right]^{\otimes [n]} \right) \otimes A \quad (3.10)$$

The derivatives with respect to C follow analogously and cross derivatives are given by

$$A_{CB} \equiv \mathcal{D}_{C^T} \{ \mathcal{D}_{B^T} \{A\} \} \equiv \mathcal{D}_{C^T B^T} \{A\} \equiv \left[\frac{\partial}{\partial c_1} \quad \dots \quad \frac{\partial}{\partial c_s} \right] \otimes \left[\frac{\partial}{\partial b_1} \quad \dots \quad \frac{\partial}{\partial b_r} \right] \otimes A \quad (3.11)$$

Additional details can be found in the Appendix. Successive differentiation of f to the desired order of approximation is a mechanical application of the following theorem

Theorem 3.2.3. *A Multidimensional Calculus*

For the matrix-valued functions F , G , A , and H and vector-valued functions J and C there exists an operator \mathcal{D}_{x^T} indicating differentiation with respect to the transpose of the column vector x . Unless indicated otherwise, all matrices and vectors are understood to be functions of the $s \times 1$ column vector B and we leave this dependency implicit.

1. *Matrix Product Rule:* $\mathcal{D}_{B^T} \left\{ \begin{smallmatrix} F & G \\ p \times q & q \times u \end{smallmatrix} \right\} = F_B \left(\begin{smallmatrix} I & \\ s \times s & \end{smallmatrix} \otimes G \right) + F G_B$
2. *Matrix Chain Rule:* $\mathcal{D}_{B^T} \left\{ \begin{smallmatrix} A & (C) \\ p \times q & u \times 1 \end{smallmatrix} \right\} = A_C \left(\begin{smallmatrix} C_B & \\ & I \\ q \times q & \end{smallmatrix} \right)$
3. *Kronecker Product Rule:* $\mathcal{D}_{B^T} \left\{ \begin{smallmatrix} F & \otimes & H \\ p \times q & & u \times v \end{smallmatrix} \right\} = F_B \otimes H + (F \otimes H_B) \left(\begin{smallmatrix} K_{q,s} & \\ & I \\ q \times s & & v \times v \end{smallmatrix} \right)$,
where $K_{q,s}$ is a $qs \times qs$ commutation matrix (see Magnus and Neudecker (1979))

where $F_B \equiv \mathcal{D}_{B^T} F$ etc. has been used as abbreviated notation to minimize clutter.

Proof. See Appendix. □

By adapting the abbreviated notation from above and writing $y_{\sigma^{n_{i_1 i_2 \dots i_m}}}$ as the partial derivative, evaluated at the deterministic steady state, of y with respect to σ for n times and with respect to $\epsilon_{t-i_1}^T, \epsilon_{t-i_2}^T, \dots, \epsilon_{t-i_m}^T$, we can then write the M -th order Taylor approximation of the policy function (3.2) using the following

Corollary 3.2.4. *An M -th order Taylor Approximation of (3.2) is written as*

$$y_t = \sum_{m=0}^M \frac{1}{m!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \dots \sum_{i_m=0}^{\infty} \left[\sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^{n_{i_1 i_2 \dots i_m}}} \sigma^n \right] (\epsilon_{t-i_1} \otimes \epsilon_{t-i_2} \otimes \dots \otimes \epsilon_{t-i_m}) \quad (3.12)$$

Proof. See Appendix. □

This infinite dimensional Taylor approximation, or finite Volterra series with kernels $[\sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^n i_1 \dots i_m} \sigma^n]$,¹² directly maps the exogenous innovations to endogenous variables up the M -th order. The kernels at m collects all the coefficients associated with the m 'th fold Kronecker products of exogenous innovations i_1, i_2, \dots and i_m periods ago. For a given set of indices, i_1, i_2, \dots and i_m , the sum over n , gathering terms in powers of the perturbation parameter σ , corrects the kernel for uncertainty up to the n -th order, thereby enabling a classification of the contributions of uncertainty to the model. That is, we can first decompose the Volterra series into kernels associated with the order of approximation in the state space itself—the zeroth kernel being constants, the first order kernel being linear in the space of the history of innovations, the second being quadratic in the same, etc. Thereafter, we can decompose each of the kernels into successively higher order corrections for uncertainty, according to the order in σ — y_{σ^n} is the n 'th order correction for uncertainty of the zeroth order kernel, $y_{\sigma^n i_1}$ the n 'th order correction for uncertainty of the first order kernel, $y_{\sigma^n i_1 i_2}$ the n 'th order correction for uncertainty of the second order kernel, and so on.

For a different perspective, observe that moving from an $M - 1$ 'th to M 'th order approximation in (A.1) comprises two changes: (i) adding a higher order kernel and (ii) opening up all existing kernels to a higher order correction for uncertainty.¹³

$$\sum_{m=0}^M \frac{1}{M!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \dots \sum_{i_m=0}^{\infty} \left[\frac{1}{(M-m)!} y_{\sigma^{M-m} i_1 i_2 \dots i_m} \sigma^{M-m} \right] (\varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \otimes \dots \otimes \varepsilon_{t-i_m}) \quad (3.13)$$

The difference can be written compactly despite the two changes, as change (i) is an M 'th order kernel with a zeroth order correction for uncertainty (for $m = M$ above, $y_{\sigma^{M-m} i_1 i_2 \dots i_m} \sigma^{M-m} = y_{\sigma^0 i_1 i_2 \dots i_m} \sigma^0 = y_{i_1 i_2 \dots i_m}$). From (ii) comes then additionally a first order correction for uncertainty in the $M - 1$ 'th order kernel, a second order uncertainty correction for the $M - 2$ 'th kernel and so on up to the M 'th order correction for uncertainty in the constant or zeroth order kernel. The uncertainty correction at a given order directly depends on the moments of future shocks at each order and so (ii) can be interpreted as successively opening each kernel up to higher moments in the distribution of future shocks, while (i) maintains the standard Taylor notion of moving to a higher order polynomial (captured by the kernels in our Volterra series).

It is instructive to consider the case of $M = 2$ (the second-order approximation)

¹²See section 3.4 for a discussion of the convergence of these infinite sums.

¹³We are grateful to Michael Burda for suggesting this interpretation. A similar interpretation can be found in Judd and Mertens (2013b) for univariate expansions of the state space representation.

given by

$$y_t = \bar{y} + y_\sigma \sigma + \frac{1}{2} y_{\sigma^2} \sigma^2 + \sum_{i=0}^{\infty} (y_i + y_{\sigma,i} \sigma) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{ji} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (3.14)$$

Here, \bar{y} , the policy function evaluated at the deterministic steady state, represents the rest point in the absence of uncertainty regarding future shocks. The terms $\sum_{i=0}^{\infty} y_i \varepsilon_{t-i}$ and $\frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{ji} (\varepsilon_{t-j} \otimes \varepsilon_{t-i})$ capture the first and second order responses to realized shocks in the absence of uncertainty regarding future shocks. The constant term has two uncertainty corrections, $y_\sigma \sigma$ and $\frac{1}{2} y_{\sigma^2} \sigma^2$ the first and second order corrections for uncertainty respectively, leading to the second order accurate stochastic steady state. At second order, $\sum_{i=0}^{\infty} y_{i\sigma} \sigma \varepsilon_{t-i}$ is the first order correction for uncertainty concerning future shocks of the first order response to the history of shocks. The first order corrections for uncertainty will turn out to be zero in this case, a familiar result from state space analyses.¹⁴ In the next section, we provide explicit derivations to third order.¹⁵

3.3 Numerical Solution of the Perturbation Approximation

In this section, we lay out the method for solving for the coefficients of the approximated solution. First order terms follow from methods well known in the literature and, as with state space methods, higher order terms solve linear equations given terms from lower orders, with terms linear in the perturbation parameter being zero. In contrast to state space methods, the systems of equations at all orders of approximation are systems of difference equations with identical homogenous components, enabling the stability from the first order to be passed on to higher orders. We rule out unit roots in the first order approximation along with the standard saddle point assumption to ensure the boundedness of uncertainty corrections to constants.

We proceed as follows.¹⁶ Inserting the policy functions (3.2), (3.3), and (3.4) along with the analogous representation, (3.5), for u_t into the model, (4.22), yields

$$0 = E_t \left[f(y^-(\sigma, \varepsilon_{t-1}, \varepsilon_{t-2}, \dots), y(\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots), y^+(\sigma, \tilde{\varepsilon}_{t+1}, \varepsilon_t, \varepsilon_{t-1}, \dots), u(\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots)) \right] \quad (3.15)$$

a function with arguments $\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots$. At each order of approximation, we take

¹⁴See Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), and Kim et al. (2008).

¹⁵See Andreasen (2012a) and Ruge-Murcia (2012) for notable extensions of Schmitt-Grohé and Uribe's (2004) method to third order; their third order derivatives fill pages, highlighting the advantage of our notation.

¹⁶See Anderson et al. (2006, p. 9) for a similar outline in their state space context.

the collection of derivatives of f from the previous order (for the first order, we start with f itself) and

1. differentiate the derivatives of f from the previous order with respect to all their arguments
2. evaluate the partial derivatives of f and of y at the deterministic steady state
3. apply the expectations operator and evaluate using the given moments
4. set the resulting expression to zero and solve for the unknown partial derivatives of y .

The partial derivatives of y , obtained in step (4) at each order, constitute the unknown coefficients of the Taylor/Volterra approximation of the policy function y . They are numeric and used again in step (2) of the next higher order. This introduces the potential for the compounding of numerical errors as we move to higher orders as highlighted by Anderson et al. (2006). The set of derivatives of f obtained in step (1), however, are symbolic at each order, limiting the source for potential error compounding to the partial derivatives of the policy function.

3.3.1 First Order Approximation

We are seeking the first order approximation of the policy function (3.2), evaluated at the deterministic steady state, \bar{x} , of the form

$$y_t = \bar{y} + y_\sigma \sigma + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} \quad (3.16)$$

The task at hand is to pin down the partial derivatives, y_σ and y_i . Even in the first order case, the problem is infinite dimensional owing to the infinite moving average representation of the solution. As explained by Taylor (1986, p. 2003), the original stochastic difference equations in y_t become deterministic difference equations in the moving average coefficients of y_t .

To determine y_i , we differentiate f in (3.15) with respect to some ε_{t-i}

$$\mathcal{D}_{\varepsilon_{t-i}^T} f = f_x x_i \quad (3.17)$$

Evaluating this at the deterministic steady state, \bar{x} , and setting its expectation to zero yields

$$E_t(\mathcal{D}_{\varepsilon_{t-i}^T} f) \Big|_{\bar{x}} = f_{y^-} y_{i-1} + f_{y_i} y_i + f_{y^+} y_{i+1} + f_u u_i = 0 \quad (3.18)$$

for $i = 0, 1, \dots$, with $y_{-1} = 0$

a second order linear deterministic difference equation in y_i , the derivatives of the vector function y with respect to its $i - 1$ 'th ε vector argument. That is, y_i contains the linear moving average coefficients of y_i with respect to the elements of ε_{t-i} . Equation (3.18) is an inhomogeneous version of Anderson and Moore's (1985) saddle point problem, solved in detail by Anderson (2010).

We make two assumptions regarding the difference equation system (3.18).

Assumption 3.3.1. *Saddle stability*

Of the $2ny$ $z \in \mathbb{C}$ such that $\det(f_{y^-} + f_y z + f_{y^+} z^2) = 0$, there are exactly ny with $|z| < 1$.

Assumption 3.3.2. *No unit roots*

There is no $z \in \mathbb{C}$ with $|z| = 1$ and $\det(f_{y^-} + f_y z + f_{y^+} z^2) = 0$

The first assumption is standard, fulfilling the Blanchard and Kahn (1980) condition. The second has been found in other analyses, e.g., Klein (2000), and here ensures the solvability of terms homogenous in σ — i.e., uncertainty corrections to the constant. Intuitively from the state space perspective, unit roots must be ruled out to allow the state space solution to be inverted, yielding the moving average representation we work with. Failing this, initial conditions on the endogenous variables would not vanish and any constant corrections would fail to converge when solving out back into the infinite past.

Anderson's (2010) method can be applied under our assumptions 3.3.1 and 3.3.2 along with the first order linear autoregressive u_t (i.e., $u_i = N^i$),¹⁷ delivering the unique stable solution to (3.18)

$$y_i = \alpha y_{i-1} + \beta_1 u_i, \text{ with } y_{-1} = 0 \quad (3.19)$$

a convergent recursion from which we can recover the linear moving average terms or y_i 's.¹⁸

To determine y_σ , we differentiate f in (3.15) with respect to σ

$$\mathcal{D}_\sigma f = f_x \mathcal{D}_\sigma x, \text{ where } \mathcal{D}_\sigma x = x_\sigma + x_{\tilde{\varepsilon}} \varepsilon_{t+1} \quad (3.20)$$

Evaluating this at \bar{x} and setting its expectation to zero yields

$$E_t(\mathcal{D}_\sigma f) \Big|_{\bar{x}} = (f_{y^-} + f_y + f_{y^+}) y_\sigma = 0 \quad (3.21)$$

¹⁷Alternatively, one can apply Klein's (2000) QZ algorithm to this deterministic approach, noting, as discussed by Meyer-Gohde (2010, pp. 986-987), that this is a deterministic saddle point problem in the moving-average coefficients and not a stochastic saddle-point problem in the endogenous variables themselves.

¹⁸We have tacitly assumed that this solution exists, see Anderson (2010, p. 483) for the details. In Klein's (2000) notation, Z_{11} of the QZ decomposition must be invertible, the added proviso of translatability.

as $E_t(\varepsilon_{t+1}) = 0$. From assumption 3.3.2, it follows that $\det(f_{y^-} + f_y + f_{y^+}) \neq 0$ and hence

$$y_\sigma = 0 \quad (3.22)$$

The first order correction of the constant for uncertainty is zero, analogous to the result of Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004).¹⁹ This result carries over to our moving average by ruling out unit roots to ensure the invertibility of the state space representation. The result itself reflects the fact that opening the expansion to a moment of the future distribution of shocks will change nothing if this moment ($E_t[\varepsilon_{t+1}]$) is exactly zero.

Gathering the results of this section, the first order approximation of the policy function (3.2), which can be thought of as an extension of Muth (1961), Taylor (1986), and others, reduces to

$$y_t = \bar{y} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} \quad (3.23)$$

Certainty equivalence at first order is reflected by the independence of (3.23) from σ .

3.3.2 Second Order Approximation

Taking the first order results as given, we now move on to the second order approximation of the policy function (3.2) evaluated at the deterministic steady state, \bar{x} , of the form

$$y_t = \bar{y} + \frac{1}{2} y_{\sigma^2} \sigma^2 + \sum_{i=0}^{\infty} (y_i + y_{\sigma,i} \sigma) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (3.24)$$

The task is to pin down the second derivatives of the y function. The equations in $y_{j,i}$ and $y_{\sigma,i}$ are difference equations with homogenous components identical to those in (3.18), with the equation in $y_{\sigma,i}$ homogenous in accordance with Schmitt-Grohé and Uribe (2004) and others. The no-unit-root assumption is crucial in solving for y_{σ^2} , preventing this constant correction for uncertainty induced by the potential for future shocks from becoming arbitrarily large.

We first differentiate (3.17) with respect to ε_{t-j} , delivering $y_{j,i}$, the derivatives of y with respect to all pairs of ε_{t-i} and ε_{t-j} . As in Judd (1998, p. 477), the resulting system is linear in $y_{j,i}$

$$\mathcal{D}_{\varepsilon_{t-j}^T \varepsilon_{t-i}^T} f = f_{x^2}(x_j \otimes x_i) + f_x x_{j,i} \quad (3.25)$$

¹⁹See also Lan and Meyer-Gohde (2012).

Evaluating at the deterministic steady state and setting its expectation to zero yields

$$E_t(\mathcal{D}_{\varepsilon_{t-i}^T} f) \Big|_{\bar{x}} = f_{y^-} y_{j-1,i-1} + f_y y_{j,i} + f_{y^+} y_{j+1,i+1} + f_{x^2} (x_j \otimes x_i) = 0 \quad (3.26)$$

for $j, i = 0, 1, \dots$, with $y_{j,i} = 0$, for $j, i < 0$

a second order linear deterministic difference equation in $y_{j,i}$. The coefficients on the homogeneous components of the foregoing and (3.18) are identical. The inhomogeneous components have a first order Markov representation (see the shifting and transition matrices defined in the Appendix) in the Kronecker product of the first order coefficients.²⁰ The resulting expression is

$$f_{y^-} y_{j-1,i-1} + f_y y_{j,i} + f_{y^+} y_{j+1,i+1} + f_{x^2} (\gamma_1 \otimes \gamma_1) (S_j \otimes S_i) = 0 \quad (3.27)$$

for $j, i = 0, 1, \dots$, with $y_{j,i} = 0$, for $j, i < 0$

The stable solution of the foregoing, analogously to the first order, takes the form

$$y_{j,i} = \alpha y_{j-1,i-1} + \beta_2 (S_j \otimes S_i), \text{ with } y_{j,i} = 0, \forall j, i < 0 \quad (3.28)$$

Note that α in this solution is known. It is the same uniquely stable α as in the first order solution (3.19) due to the fact that the system (3.26) and (3.18) have identical homogeneous components. To determine β_2 , we substitute (3.28) in (3.26), using the shifting matrices and matching coefficients

$$(f_y + f_{y^+} \alpha) \beta_2 + f_{y^-} \beta_2 (\delta_1 \otimes \delta_1) = -f_{x^2} (\gamma_1 \otimes \gamma_1) \quad (3.29)$$

This is a type of Sylvester equation, solved in detail by Kamenik (2005).

Next we pin down $y_{\sigma,i}$, the second derivatives of the y function with respect to ε_{t-i} and σ sequentially, by differentiating (3.17) with respect to σ . The resulting linear system is

$$\mathcal{D}_{\sigma \varepsilon_{t-i}^T} f = f_{x^2} (\mathcal{D}_{\sigma} x \otimes x_i) + f_x \mathcal{D}_{\sigma} x_i, \text{ where } \mathcal{D}_{\sigma} x_i = x_{\sigma,i} + x_{\varepsilon,i} (\varepsilon_{t+1} \otimes I_{ne}) \quad (3.30)$$

Note that $\mathcal{D}_{\varepsilon_{t-i}^T} \sigma f$ is simply equal to $\mathcal{D}_{\sigma \varepsilon_{t-i}^T} f$.²¹ Evaluating (3.30) at \bar{x} , taking expectations, noting that $y_{\sigma} = 0$, and setting the resulting expression to zero yields

$$E_t(\mathcal{D}_{\sigma \varepsilon_{t-i}^T} f) \Big|_{\bar{x}} = f_{y^-} y_{\sigma,i-1} + f_y y_{\sigma,i} + f_{y^+} y_{\sigma,i+1} = 0 \quad (3.31)$$

for $i = 0, 1, \dots$, with $y_{\sigma,-1} = 0$

²⁰Thus, our nonlinear moving average solution parallels nonlinear state space solutions in a manner analogous to the linear case, where the recursion is in the coefficients as opposed to the variables themselves. Instead of products of the state variables entering into the solution, we have products of the first order coefficients.

²¹ Although the derivative operator \mathcal{D} works on Kronecker products (i.e. $\mathcal{D}_{\sigma \varepsilon_{t-i}^T} = \mathcal{D}_{\sigma \otimes \varepsilon_{t-i}^T} f$) that are not generally commutative, σ is a scalar and, thus, commutation is preserved. Accordingly, the order in which derivatives with respect to σ appear is inconsequential as it is a scalar and we choose to have the σ 's appear first.

The unique stable solution takes the form

$$y_{\sigma,i} = \alpha y_{\sigma,i-1}, \text{ for } i = 0, 1, \dots, \text{ with } y_{\sigma,-1} = 0 \quad (3.32)$$

as the system at hand is identical to the homogenous component of the first order system (3.18). Combined with the initial condition $y_{\sigma,-1} = 0$, the foregoing delivers

$$y_{\sigma,i} = 0, \text{ for } i = 0, 1, \dots \quad (3.33)$$

Again, we confirm that terms with a first order uncertainty correction are zero.

Finally, to determine y_{σ^2} , the second derivative of the y function with respect to σ , we differentiate (3.20) with respect to σ . The resulting linear system is

$$\mathcal{D}_{\sigma^2} f = f_{x^2}(\mathcal{D}_{\sigma} x \otimes \mathcal{D}_{\sigma} x) + f_x \mathcal{D}_{\sigma^2} x, \text{ where } \mathcal{D}_{\sigma^2} x = x_{\sigma^2} + 2x_{\sigma, \tilde{\varepsilon}} \varepsilon_{t+1} + x_{\tilde{\varepsilon}^2}(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) \quad (3.34)$$

Evaluating this at \bar{x} and setting its expectation to zero yields

$$E_t(\mathcal{D}_{\sigma^2} f) \Big|_{\bar{x}} = [f_{y^+} y_{0^2} + f_{y^+2}(y_0 \otimes y_0)] E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) + (f_{y^-} + f_y + f_{y^+}) y_{\sigma^2} = 0 \quad (3.35)$$

therefore we can recover y_{σ^2} by

$$y_{\sigma^2} = -(f_{y^-} + f_y + f_{y^+})^{-1} [f_{y^+} y_{0^2} + f_{y^+2}(y_0 \otimes y_0)] E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) \quad (3.36)$$

By assumption, the second moment of the exogenous shock, $E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1})$, is given.

As the model approaches a unit root from below, this uncertainty correction becomes unbounded. This result is novel, giving additional meaning to assumption 3.3.2: from a state space perspective, the correction for uncertainty will be accumulated forward starting from the deterministic steady state; if the state space contains a unit root, this accumulated correction will increase without bound and there will be no finite stochastic steady state.

In sum, the second order approximation of the policy function (3.2) takes the form

$$y_t = \bar{y} + \frac{1}{2} y_{\sigma^2} \sigma^2 + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (3.37)$$

In contrast to the first order approximation, (3.37) does depend on σ , with the term $\frac{1}{2} y_{\sigma^2}$ correcting the deterministic steady state for uncertainty regarding future shocks. As σ goes to 1 and we transition to uncertain model, the rest point of the solution transitions from the deterministic steady state \bar{y} to the second order approximation of the stochastic steady state $\bar{y} + \frac{1}{2} y_{\sigma^2} \sigma^2$. As we are interested in this uncertain version, setting σ to one in (3.37) gives the second order approximation

$$y_t = \bar{y} + \frac{1}{2} y_{\sigma^2} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (3.38)$$

3.3.3 Third Order and Higher Approximations

Given the results from lower orders, including that terms linear in the perturbation parameter are zero, the third order approximation of the y function we are seeking takes the form

$$y_t = \bar{y} + \frac{1}{2}y_{\sigma^2}\sigma^2 + \frac{1}{6}y_{\sigma^3}\sigma^3 + \sum_{i=0}^{\infty} \left(y_i + \frac{1}{2}y_{\sigma^2,i}\sigma^2 \right) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (y_{j,i} + y_{\sigma,j,i}\sigma) (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) + \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k,j,i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (3.39)$$

The task at hand is to pin down the third derivatives of the y function, including $y_{k,j,i}$, $y_{\sigma^2,i}$, $y_{\sigma,j,i}$ and y_{σ^3} . As the computation of these derivatives largely resembles that of the second derivatives in the previous section, we relegate the details to the Appendix and focus on the results here.

To determine $y_{k,j,i}$, we differentiate (3.25) with respect to some shocks ε_{t-k} , delivering the third derivatives of the y function with respect to all triplets of the shocks. The resulting system, evaluated at \bar{x} and in expectation, of equations is a linear deterministic second order difference equation in $y_{k,j,i}$. The homogeneous components in (B.15) are identical to those in (3.18) and (3.26) and the inhomogeneous components can again be rearranged into a first order Markov form

$$E_t(\mathcal{D}_{\varepsilon_{t-k}^T \varepsilon_{t-j}^T \varepsilon_{t-i}^T} f) \Big|_{\bar{x}} = f_{y^-} y_{k-1,j-1,i-1} + f_y y_{k,j,i} + f_{y^+} y_{k+1,j+1,i+1} + \begin{bmatrix} f_{x^3} & f_{x^2} & f_{x^2} & f_{x^2} \end{bmatrix} \gamma_3 S_{k,j,i} = 0$$

for $k, j, i = 0, 1, \dots$, with $y_{k,j,i} = 0$, for $k, j, i < 0$ (3.40)

The unique stable solution of the foregoing, analogously to lower orders, takes the form

$$y_{k,j,i} = \alpha y_{k-1,j-1,i-1} + \beta_3 S_{k,j,i}, \text{ with } y_{k,j,i} = 0, \text{ for } k, j, i < 0 \quad (3.41)$$

and β_3 can be solved for by, again, formulating an appropriate Sylvester equation.

To determine $y_{\sigma,j,i}$, we differentiate (3.25) with respect to σ , evaluate at \bar{x} , take expectations, set the resulting expression to zero, and recall the results from lower orders, yielding

$$E_t(\mathcal{D}_{\sigma \varepsilon_{t-j}^T \varepsilon_{t-i}^T} f) \Big|_{\bar{x}} = f_{y^-} y_{\sigma,j-1,i-1} + f_y y_{\sigma,j,i} + f_{y^+} y_{\sigma,j+1,i+1} = 0 \quad (3.42)$$

for $j, i = 0, 1, \dots$, with $y_{\sigma,j,i} = 0$, for $j, i < 0$

or, again confirming that terms with a first order uncertainty are zero,

$$y_{\sigma,j,i} = 0, \text{ for } j, i = 0, 1, \dots \quad (3.43)$$

To determine $y_{\sigma^2,i}$, we differentiate (3.30) with respect to σ , evaluate at \bar{x} , take expectations, set the resulting expression to zero, and recall the results from lower

orders, yielding

$$\begin{aligned}
E_t(\mathcal{D}_{\sigma^2 \varepsilon_{t-i}^T} f) \Big|_{\bar{x}} &= f_{x^3} \{ [(x_{\tilde{\varepsilon}} \otimes x_{\tilde{\varepsilon}}) E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1})] \otimes x_i \} \\
&\quad + 2f_{x^2}(x_{\tilde{\varepsilon}} \otimes x_{\tilde{\varepsilon}}) [E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) \otimes I_{ne}] \\
&\quad + f_{x^2} \{ (x_{\sigma^2} \otimes x_i) + [x_{\tilde{\varepsilon}^2} E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1})] \otimes x_i \} \\
&\quad + f_x \{ x_{\sigma^2, i} + x_{\tilde{\varepsilon}^2, i} [E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) \otimes I_{ne}] \} \\
&= 0, \text{ for } i = 0, 1, \dots, \text{ with } y_{-1} = 0
\end{aligned} \tag{3.44}$$

which is still a second order deterministic difference equation. The homogeneous components are packed in $x_{\sigma^2, i}$ and they are identical to those in (3.18) and (3.26). The inhomogeneous components can again be rearranged to have a first order Markov representation and the unique stable solution of the foregoing takes the form

$$y_{\sigma^2, i} = \alpha y_{\sigma^2, i-1} + \beta_{\sigma} S_i, \text{ with } y_{\sigma^2, -1} = 0 \tag{3.45}$$

where β_{σ} can be solved for by, again, formulating an appropriate Sylvester equation.

To determine y_{σ^3} , we differentiate (3.34) with respect to σ , evaluate at \bar{x} , take expectations, set the resulting expression to zero, and recall the results from lower orders, yielding

$$\begin{aligned}
E_t(\mathcal{D}_{\sigma^3} f) \Big|_{\bar{x}} &= f_{x^3} [(x_{\tilde{\varepsilon}} \otimes x_{\tilde{\varepsilon}} \otimes x_{\tilde{\varepsilon}}) E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})] \\
&\quad + 3f_{x^2} [(x_{\tilde{\varepsilon}^2} \otimes x_{\tilde{\varepsilon}}) E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})] \\
&\quad + f_x [y_{\sigma^3} + x_{\tilde{\varepsilon}^3} E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})] = 0
\end{aligned} \tag{3.46}$$

as the third moment of ε_t is assumed given, $E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})$ is known. Recovering y_{σ^3} from the foregoing is straightforward under the assumption 3.3.2. When ε_t is normally distributed,²² however, $E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1}) = 0$. Hence

$$y_{\sigma^3} = 0 \tag{3.47}$$

Combining, the third order approximation of the policy function (3.2) takes the form

$$\begin{aligned}
y_t &= \bar{y} + \frac{1}{2} y_{\sigma^2} \sigma^2 + \sum_{i=0}^{\infty} \left(y_i + \frac{1}{2} y_{\sigma^2, i} \sigma^2 \right) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j, i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \\
&\quad + \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k, j, i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i})
\end{aligned} \tag{3.48}$$

Again in contrast to the first order approximation, (3.48) does depend on σ , with the term $\frac{1}{2} y_{\sigma^2}$ correcting the deterministic steady state for uncertainty as in the second order approximation (3.37), but now with $\frac{1}{2} y_{\sigma^2, i} \sigma^2$ correcting the first order kernel for uncertainty; i.e., as σ goes from 0 to 1 and we transition from the certain to uncertain

²²As is the case in Dynare, see Adjemian et al. (2011). See also Andreasen (2012a) for third order DSGE state space perturbations with skewed distributions.

model, we incorporate the additional possibility of a time-varying correction for uncertainty. As we are interested in the original, uncertain formulation, setting σ to one in (3.48) gives the third order approximation

$$\begin{aligned} y_t = & \bar{y} + \frac{1}{2}y_{\sigma^2} + \sum_{i=0}^{\infty} \left(y_i + \frac{1}{2}y_{\sigma^2,i} \right) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \\ & + \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k,j,i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i}) \end{aligned} \quad (3.49)$$

Higher order approximations of the policy function (3.2) can be computed using the same steps. At each order of approximation, the undetermined derivatives of the policy function will always be terms of highest order being considered, ensuring that the leading coefficient matrix is f_x . Thus, for all time varying components, the difference equations in these components will have the same homogenous representation—for non time varying components (i.e. derivatives with respect to σ only), the leading coefficient matrix f_x along with assumption 3.3.2 ensure the uniqueness of their solution. The inhomogenous elements of the difference equations in the time varying components will be composed of terms of lower order, which are necessarily constants (terms in the given moments and derivatives with respect to σ only) or products of stable recursions (time varying components of lower order). Thus, we conclude from assumption 3.3.1 that the difference equations in all time varying components are saddle stable and, hence, the stability of the first order recursion is passed to all higher orders.

3.4 Comparison with Alternative Methods

Here we compare our perturbation with a nonlinear moving average with other state space methods in the literature, namely standard DSGE perturbations, e.g. as in Jin and Judd (2002) or Schmitt-Grohé and Uribe (2004), and Lombardo's (2010) matched perturbation. These comparisons serve, firstly, to demonstrate that the boundedness that the state space methods can only guarantee after pruning directly follows from our representation of the solution as a nonlinear moving average. Secondly, Lombardo (2010) offers a stable state space solution—a matched perturbation or series expansion—by appealing to the mathematical perturbation literature,²³ by contrast, our nonlinear moving average finds its foundations—existence, representation, and

²³See Murdock (1991) and Holmes (1995).

approximation theorems—in the systems theoretical literature on Volterra series.²⁴

3.4.1 Standard DSGE Perturbation: Boundedness and Pruning

Given the (local) stability of the underlying nonlinear system, we are interested in the stability properties of approximations. We will now show that stability is guaranteed at every order for our nonlinear moving average, provided that the first order is stable. This contrasts with standard DSGE perturbations that must be pruned²⁵ to ensure stability at higher order, see Kim et al. (2008) and Den Haan and De Wind (2012b).

We begin with the stability of our nonlinear moving average. The finite Volterra series (A.1) is bounded-input bounded-output stable (BIBO) if its kernels are bounded

Theorem 3.4.1 (BIBO Stability of Finite Volterra Series with Bounded Kernels). *If*

$$\sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left\| \sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^n i_1 \dots i_m} \sigma^n \right\| < \infty, \text{ for } m = 0, 1, \dots, M \quad (3.50)$$

then $\sup_{-\infty \leq t \leq \infty} \|y_t\| < \infty$ *if* $\sup_{-\infty \leq t \leq \infty} \|\varepsilon_t\| < \infty$.

Proof. Define $a_m \equiv \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \left\| \sum_{n=0}^{M-m} \frac{1}{n!} y_{\sigma^n i_1 \dots i_m} \sigma^n \right\|$ and $\|\varepsilon\| \equiv \sup_{-\infty \leq t \leq \infty} \|\varepsilon_t\| < \infty$, then $\|y_t\| \leq a_0 + a_1 \|\varepsilon\| + a_2 \|\varepsilon\|^2 + \dots + a_M \|\varepsilon\|^M < \infty$ for all t , whence the boundedness of the sequence of y_t follows. See Borys (2001, ch. 1.6) for details. \square

See Sandberg (1990) for the case of scalar y_t and ε_t . Hamilton (1994, pp. 69–70) contains a proof for the linear ($M = 1$) case; i.e., that the boundedness of the first order kernel is necessary and sufficient for the stability of the (linear) infinite moving average. In contrast to this linear case, as Sandberg (1990) points out, the equivalent conditions on the kernels in theorem 3.4.1 in nonlinear representations (Volterra series with $M > 1$) are merely sufficient.

Intuitively and as pointed out in section 3.3, the kernels at every order of approximation are given by linear difference equations with the same homogenous component as at first order with inhomogenous terms that are (Kronecker) products of lower order recursions. As there is a unique stable solution to the homogenous component following assumption 3.3.1 and the first order transition matrix of the exogenous process u_t , N , has all its eigenvalues inside the unit circle, the recursions that describe the kernels

²⁴A detailed comparison in terms of accuracy and equivalence of the different pruning algorithms offered in the literature is beyond the scope of this paper. This interested reader is directed to Lan and Meyer-Gohde (2013a).

²⁵That is, the removal of terms of a higher order than the approximation that arise upon iteration of the state space policy function.

are exponentially stable, i.e., the stability properties of the first order kernel are passed on to higher orders. With all moments of ε_t up to the order of approximation finite and assumption 3.3.2 guaranteeing solvability, the terms in the zeroth kernel (i.e., deterministic steady state and constant risk corrections) exist and are finite. Thus, it follows straightforwardly from theorem 3.4.1 that the Volterra expansion (A.1) is BIBO stable under assumptions 3.3.1 and 3.3.2.

In contrast to our infinite nonlinear moving average policy function (3.2), the DSGE perturbation literature generally seeks solutions in a state space form²⁶

$$y_t = y(\sigma, z_t), \text{ where } z_t = \begin{bmatrix} y_{t-1} \\ \varepsilon_t \end{bmatrix} \quad (3.51)$$

This representation leads to a different Taylor approximation than our finite Volterra series

Proposition 3.4.2 (State Space Taylor Series Approximation). *A Taylor series approximation of $y_t = y(\sigma, z_t)$ at the deterministic steady state can be written*

$$y_t = \sum_{j=0}^M \frac{1}{j!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j} \sigma^i \right] (z_t - \bar{z})^{\otimes [j]} \quad (3.52)$$

Proof. See Lan and Meyer-Gohde (2012). □

Notice that (3.51)—and consequentially, (3.52)—map $\{\sigma, \varepsilon_t, y_{t-1}\} \rightarrow y_t$ instead of $\{\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots\} \rightarrow y_t$ as (3.2)—and consequentially, (A.1)—do. This is not contradictory, as we can iterate on the mapping $\{\sigma, \varepsilon_t, y_{t-1}\} \rightarrow y_t$ to generate the mapping $\{\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots\} \rightarrow y_t$, provided the system is stable.²⁷

Kim et al. (2008) among others in the literature have demonstrated that the dynamics of higher order local approximations of the state space policy function (3.51) need not be stable, even when the linear approximation is. The same arguments as above will highlight the issue: the largest element in y_t can be bounded by

$$\|y_t - \bar{y}\| \leq b_0 + b_1 \|\hat{z}_t\| + b_2 \|\hat{z}_t\|^2 + \dots + b_M \|\hat{z}_t\|^M < \infty \quad (3.53)$$

where $b_0 = \|\sum_{i=1}^M \frac{1}{i!} y_{z^j} \sigma^i\|$, $b_j = \frac{1}{j!} \|\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j} \sigma^i\|$, for $j > 0$, and $\hat{z}_t = z_t - \bar{z}$. Now, let $\|y_t - \bar{y}\| = Y$, then $\|\hat{z}_{t+1}\| \leq Y$ and

$$\|y_{t+1} - \bar{y}\| \leq b_0 + b_1 \|Y\| + b_2 \|Y\|^2 + \dots + b_M \|Y\|^M \quad (3.54)$$

with $\|y_{t+s} - \bar{y}\|$ bounded by terms of the order $\|Y\|^{M^s}$ when this process is continued. That the order in $\|Y\|^{M^s}$ increases in s , i.e., when the approximative state space policy

²⁶This definition implicitly rules out serial correlation in u_t , seemingly forcing $u_t = \varepsilon_t$. By extending the the vector y_t to include u_t , however, the original problem statement with exogenous serial correlation can be restored.

²⁷Assumptions 3.3.1 and 3.3.2 ensure the local stability of (3.51) and permit this iteration locally. Globally, additional assumptions, such as fading memory, are necessary, see the next subsection.

function (3.52) is iterated forward reflects, as noted by Kim et al. (2008, p. 3408), (i) the accumulation of extra higher order terms and (ii) the potential for explosive times paths in endogenous variables as this bound can increase limitlessly. Of course, explosive time paths are not a certain outcome for time paths and truncation of the distribution from which exogenous shocks are drawn or the application of pruning schemes, like proposed by Kim et al. (2008), can prevent such behavior.

Whereas our nonlinear moving average policy function and its local approximation (Volterra series) are defined in space of sequences (the infinite history of shocks), the standard DSGE state space policy function and its local approximation (Taylor series) are defined in a different space that maps state variables into endogenous variables. Calculating a simulation requires iteration or successive substitution of the state space policy function, whereas a simulation is simply a single evaluation of the nonlinear moving average policy function for a particular sequence of shocks. The boundedness of time paths for endogenous variables follows automatically from our choice of approximation and no ex post pruning or truncation is needed.

3.4.2 Lombardo's (2010) Matched Perturbation: Foundations

Lombardo (2010) presents a nonlinear local method that generates stable time paths for endogenous variables with the stability from first order passed on to higher orders. Yet there are subtle differences. Firstly, he uses σ to expand from the deterministic steady state to the stochastic dynamic solution, scaling all realizations of shocks, past and present, and the distribution of future shocks with σ .²⁸ In contrast, the formulation we have used above following Jin and Judd (2002) and others uses σ to expand the deterministic dynamic solution²⁹ to the stochastic dynamic solution, scaling only the uncertainty that arises through the distribution of future shocks with σ .

Second, Lombardo (2010) derives his method from the applied mathematics literature on matched perturbations, see Murdock (1991) and Holmes (1995), providing a substantive formal basis for his method and associated assumptions. While there is perhaps no single right way to perturb, this formalism would lend the series expansion or matched perturbation method of Lombardo (2010) significant credence. Our nonlinear moving average can likewise be given a more formal basis by appealing to the systems theory literature on Volterra series. We shall now do as such and provide a representa-

²⁸ See also Den Haan and De Wind (2012b), who state in their supplemental Appendix that Lombardo's (2010) method "does not describe any transition dynamics" when $\sigma = 0$.

²⁹ Deterministic, that is, from the time t perspective of the equilibrium conditions in (4.22). The expectations conditional on time t information, render shocks, past and present, realizations of a stochastic process in contrast to future variables which remain stochastic from this time t perspective.

tion theorem and an approximation theorem, as well as proving the approximation is covariance stationary.

We begin with a preliminary assumption

Assumption 3.4.3. y_t is strongly stationary and possesses fading memory.

Borys (2001, ch. 1.7) shows the equivalence between stationarity in a output series and time invariance of an operator acting on an input sequence. Boyd and Chua's (1985) fading memory³⁰ requires that two series with close recent histories will be close to each other even if their distant pasts are arbitrarily far apart; that is, y_t is asymptotically independent of initial conditions Boyd and Chua (1985). Granger's (1995) short memory in mean, requiring the forecasting relevance of the initial information set to become irrelevant, embodies the same concept.

With these assumptions in hand, we repeat the representation theorem of Gourioux and Jasiak (2005) that shows that y_t has an infinite Volterra series representation, justifying our Volterra approach *per se*, and that y_t can be approximated with arbitrary accuracy by a finite Volterra series representation, such as our policy function approximation in (A.1).

Proposition 3.4.4. A square integrable³¹ y_t satisfying assumption 3.4.3 can be represented by

$$y_t = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \tilde{y}_{i_1 i_2 \dots i_m} (\varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \otimes \cdots \otimes \varepsilon_{t-i_m}) \quad (3.55)$$

Proof. See Gourioux and Jasiak (2005). □

With $\tilde{y}_{i_1 i_2 \dots i_m} = \sum_{n=0}^{\infty} \frac{1}{n!} y_{i_1 i_2 \dots i_m} \sigma^n$ following from the assumption of analyticity, establishing the equivalence between the foregoing and the limiting representation of (A.1). This is an analogue to the equivalence between autoregressive and moving average representations from linear, see Hamilton (1994, ch. 3), in nonlinear settings.

Proposition 3.4.5. Let y_t satisfy assumption 3.4.3, then for any bounded sequence $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ and $\iota > 0$, there exists an \hat{M} such that

$$\|y_t - \sum_{m=0}^{\hat{M}} \frac{1}{m!} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \tilde{y}_{i_1 i_2 \dots i_m} (\varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \otimes \cdots \otimes \varepsilon_{t-i_m})\| \leq \iota \quad (3.56)$$

Proof. See Boyd and Chua (1985) and Sandberg (2003). □

³⁰See Sandberg (2003) for a more recent systems theoretical overview.

³¹Gourioux and Jasiak (2005) emphasize the weakness of the assumption of square integrability.

This result ties the representation result above to the BIBO stability of our Volterra approximation in the previous subsection. With the Volterra series itself defined in the space of bounded sequences, seeking representations for bounded simulations amounts to finding an approximation to the Volterra representation in proposition 3.4.4. As an additional consequence of the BIBO stability, our Volterra approximation is covariance stationary

Corollary 3.4.6 (Covariance Stationarity of Finite Volterra Series). *Let the assumptions of theorem 3.4.1 hold. If $E(\varepsilon_t^{\otimes [n]})$ exists and is finite for all n up to $2M$, where M is the order of approximation, then the Volterra series approximation of y_t is covariance stationary.*

Proof. See Cariolaro and Di Masi (1980). □

Thus, our choice of seeking an approximation in the space of sequences allows us to appeal to the literature on Volterra series and rigourously prove the existence of this representation, the appropriateness of its approximation as a finite Volterra series as in (A.1), and the BIBO stability and covariance stationarity of this approximation given a stable solution at first order.

3.5 Stochastic Neoclassical Growth Model

In this section, we examine two versions of the stochastic neoclassical growth model to demonstrate the method. This model has been used in numerous studies comparing numerical techniques and is a natural benchmark. We begin with the special case of log preferences in consumption and full depreciation that has a known solution to illustrate the relation of the nonlinear moving average to the more familiar state space solution. We then move on to the baseline specification of Aruoba et al.'s (2006) comprehensive study with inseparable utility to foster comparability with their results. This version of the model lacks a known solution and must be approximated. Using our nonlinear moving average solution, we analyze the contributing elements to the response of the model's endogenous variables to a technology shock and highlight the features of the multidimensional kernels and impulse responses.

The model is populated by an infinitely lived representative household seeking to maximize its expected discounted lifetime utility given by

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t U(C_t, L_t) \right], \text{ with } U(C_t, L_t) = \frac{(C_t^\theta (1 - L_t)^{1-\theta})^{1-\gamma}}{1 - \gamma} \quad (3.57)$$

where C_t is consumption, L_t labor, and $\beta \in (0, 1)$ the discount factor, subject to

$$C_t + K_t = e^{Z_t} K_{t-1}^\alpha L_t^{1-\alpha} + (1 - \delta) K_{t-1} \quad (3.58)$$

where K_t is the capital stock accumulated today for productive purposes tomorrow, Z_t a stochastic productivity process, $\alpha \in [0, 1]$ the capital share, and $\delta \in [0, 1]$ the depreciation rate. Output Y_t is given by $e^{Z_t} K_{t-1}^\alpha L_t^{1-\alpha}$ and investment I_t by $K_t - (1 - \delta) K_{t-1}$. Productivity is described by

$$Z_t = \rho_Z Z_{t-1} + \varepsilon_{Z,t}, \quad \varepsilon_{Z,t} \sim \mathcal{N}(0, \sigma_Z^2) \quad (3.59)$$

with $|\rho_Z| < 1$ and $\varepsilon_{Z,t}$ the innovation with standard deviation σ_Z .

The solution is characterized by the intertemporal Euler condition equalizing the expected present-discounted utility value of postponing consumption one period to its utility value today

$$\frac{(C_t^\theta (1 - L_t)^{1-\theta})^{1-\gamma}}{C_t} = \beta E_t \left[\frac{(C_{t+1}^\theta (1 - L_{t+1})^{1-\theta})^{1-\gamma}}{C_{t+1}} (\alpha e^{Z_{t+1}} K_t^{\alpha-1} L_{t+1}^{1-\alpha} + 1 - \delta) \right] \quad (3.60)$$

and the intratemporal condition equalizing the utility cost of marginally increasing labor supply to the utility value of the additional consumption provided therewith

$$\frac{1 - \theta}{1 - L_t} = \frac{\theta}{C_t} (1 - \alpha) e^{Z_t} K_{t-1}^\alpha L_t^{-\alpha} \quad (3.61)$$

plus the budget constraint (3.58) and the technology shock (3.59). Collecting the four equations into a vector of functions, the set of equilibrium conditions can be written $0 = E_t[f(y_{t-1}, y_t, y_{t+1}, u_t)]$ where $y_t = [C_t \ K_t \ L_t \ Z_t]'$ and $u_t = [\varepsilon_{Z,t}]$.

3.5.1 Logarithmic Preferences and Complete Depreciation Special Case

We will first examine the model under log preferences and complete capital depreciation. This enables a scalar version in one endogenous variable of the method to be studied, and possesses a well-known closed-form solution for the state space policy function. We relate this to our policy function and use our resulting closed-form solution for an initial appraisal of our method.

Accordingly, let $U(C_t, L_t)$ in (3.57) be given by $\ln(C_t)$,³² normalize $L_t = 1$ and set $\delta = 1$ in (3.58). Combining (3.58) with (3.60) in this case yields

$$0 = E_t \left[(e^{Z_t} K_{t-1}^\alpha - K_t)^{-1} - \beta (e^{Z_{t+1}} K_t^\alpha - K_{t+1})^{-1} (\alpha e^{Z_{t+1}} K_t^{\alpha-1}) \right] \quad (3.62)$$

³²That is, set θ to one, subtracting an appropriate constant and extending over the removable singularity at $\gamma = 1$.

This case has a well-known closed form state space solution: $K_t = \alpha\beta e^{Z_t} K_{t-1}^\alpha$. However, we are interested in its infinite nonlinear moving average representation and guess that the logarithm of the solution is linear in the infinite history of technology innovations $\ln(K_t) = \ln(\bar{K}) + \sum_{j=0}^{\infty} b_j \varepsilon_{Z,t-j}$. Inserting the guess and the MA(∞) representation for Z_t , (3.62) is

$$1 = \alpha\beta E_t \left[\frac{1 - \exp\left(\sum_{j=0}^{\infty} (\rho^j - b_j + \alpha b_{j-1}) \varepsilon_{Z,t-j} - (1-\alpha) \ln(\bar{K})\right)}{1 - \exp\left(\sum_{j=0}^{\infty} (\rho^j - b_j + \alpha b_{j-1}) \varepsilon_{Z,t+1-j} - (1-\alpha) \ln(\bar{K})\right)} \right. \\ \left. \times \exp\left(\sum_{j=0}^{\infty} (\rho^j - b_j + \alpha b_{j-1}) \varepsilon_{Z,t} - (1-\alpha) \ln(\bar{K})\right) \right] \quad (3.63)$$

With $b_{-1} = 0$, $\bar{K} = (\alpha\beta)^{\frac{1}{1-\alpha}}$ and $b_j = \alpha b_{j-1} + \rho^j$ solve (3.63), verifying the guess.

Not surprisingly, this solution can also be deduced directly from the known state space solution. Take logs of $K_t = \alpha\beta e^{Z_t} K_{t-1}^\alpha$, yielding $\ln(K_t) = \ln(\alpha\beta) + Z_t + \alpha \ln(K_{t-1})$. Making use of the lag operator, L , and defining $\rho(L) = \sum_{j=0}^{\infty} (\rho L)^j$, the foregoing can be written as $\ln(K_t) = (1-\alpha)^{-1} \ln(\alpha\beta) + (1-\alpha L)^{-1} \rho(L) \varepsilon_{Z,t}$ and restating in levels gives

$$K_t = (\alpha\beta)^{\frac{1}{1-\alpha}} \exp\left((1-\alpha L)^{-1} \rho(L) \varepsilon_{Z,t}\right) = (\alpha\beta)^{\frac{1}{1-\alpha}} \exp\left(\sum_{j=0}^{\infty} b_j \varepsilon_{Z,t-j}\right) \quad (3.64)$$

where $b(L) = (1-\alpha L)^{-1} \rho(L) = \sum_{j=0}^{\infty} b_j L^j$ as before.

This special case offers a simple check of the numerical approach. We define $\hat{K}_t = \ln(K_t)$ and use $K_t = \exp(\hat{K}_t)$ to reexpress (3.62) as³³

$$0 = E_t \left[\left(e^{Z_t + \alpha \hat{K}_{t-1}} - e^{\hat{K}_t} \right)^{-1} - \beta \left(e^{Z_{t+1} + \alpha \hat{K}_t} - e^{\hat{K}_{t+1}} \right)^{-1} \left(\alpha e^{Z_{t+1} + (\alpha-1)\hat{K}_t} \right) \right] \quad (3.65)$$

With this reformulation, the first order expansion is the true policy rule in this special case. That is, (3.65) can be rewritten as $0 = E_t[f(y_{t-1}, y_t, y_{t+1}, u_t)]$ where $y_t = [\hat{K}_t \ Z_t]'$ and $u_t = [\varepsilon_{Z,t}]$.

To check our method, we calculate the kernels of the third order nonlinear moving average solution of (3.65) out 500 periods, following Hansen (1985) for the remaining parameters by setting $\alpha = 0.36$, $1/\beta = 1.01$, $\rho = 0.95$, and $\sigma_Z = 0.00712$. Our method successfully identifies $y_{j,i}$, $y_{k,j,i}$, and $y_{\sigma^2,i}$ as being zero and the largest absolute difference in y_i from the analytic solution was 4.3368×10^{-18} . This first check, while encouraging, is far from comprehensive. In section 3.6, additional and more meaningful measures will be examined.

³³See Fernández-Villaverde and Rubio-Ramírez (2006) for more on change of variable techniques such as this.

Table 3.1 Parameter Values for the Model of Section 3.5.2

Parameter	β	τ	θ	α	δ	ρ_Z	σ_Z
Value	0.9896	2.0	0.357	0.4	0.0196	0.95	0.007

See Aruoba et al. (2006).

3.5.2 CRRA-Incomplete Depreciation Case

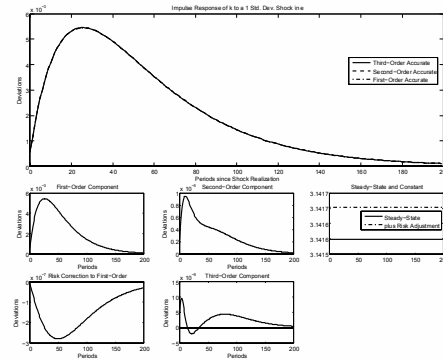
We now move to the general case of Aruoba et al. (2006). Following their parameterization, we relax the complete depreciation and log preferences of the previous section, see table 3.1. As no known closed form solution exists, we will need an approximation. We reexpress variables in logs, commensurate with a loglinear approximation for the first order approximation.

For higher-order approximations, our policy function enables impulse response analysis straightforwardly. That is, consider a shock in t to an element of ε_t , one measure³⁴ for the response of y through time to this impulse is given by the sequence $\{y_{t+s}\}_{s=0}^{\infty}$ with $\varepsilon_{t+s} = 0 \forall s$ but $s = 0$. Accordingly, $y_t = y(\sigma, \varepsilon_t, 0, 0, \dots)$, $y_{t+1} = y(\sigma, 0, \varepsilon_t, 0, \dots)$, and so forth.

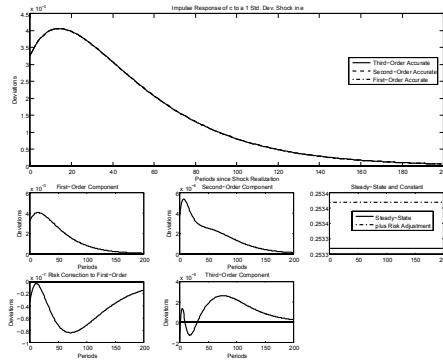
Figure 3.1 depicts the impulse responses and their contributing components from the kernels of different orders for capital, consumption, and labor to a positive, one standard deviation shock in $\varepsilon_{Z,t}$.³⁵ The upper panel displays the impulse responses at first, second, and third order as deviations from their respective (non)stochastic steady states (themselves in the middle right panel) and the first feature to notice is that they are indistinguishable to the eye. This is not surprising, as it is well known that the neoclassical growth model is nearly loglinear. In the middle column of panels in the lower half of each figure, the contributions to the total impulse responses from the second and third order kernels $y_{i,i}$ and $y_{i,i,i}$ are displayed. These components display multiple ‘humps’ to either side of the ‘hump’ in the first order component (upper left panel) in accordance with the artifact of harmonic distortion discussed in Priestly (1988, p. 27).

³⁴ Note that we are assuming that $y_{t-j} = y(\sigma, 0, 0, \dots)$, $\forall j > 0$. Fernández-Villaverde et al. (2011b), for example, examine the responses starting from the mean of the ergodic distribution instead of our stochastic steady state. In a nonlinear environment, the model must be constantly buffeted with shocks to maintain variables around the ergodic mean, imparting the response to any single impulse from the ergodic mean with a deterministic transition to the stochastic steady state. Our measure eliminates such deterministic trends in impulse responses.

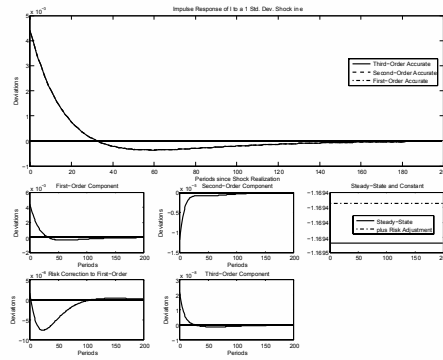
³⁵ In terms of Koop et al. (1996), we are assuming a particular history of shocks (namely the infinite absence thereof—such interaction will be addressed later), are examining a particular shock realization (positive, one standard deviation: due to the nonlinearity, asymmetries and the absence of scale invariance are a potential confound), and ignoring distributional composition issues by examining a realization of a single structural shock (here, there is only one shock, so this is moot anyway).



(a) Capital



(b) Consumption



(c) Labor

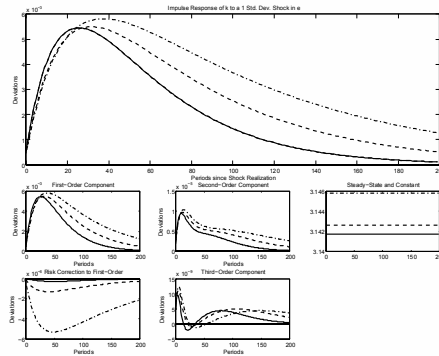
Fig. 3.1 Impulse Responses to a Technology Shock, Model of Section 3.5.2

The second order contributions of capital and consumption are positive and that of labor is negative. This reflects the combination of a precautionary reaction and nonlinear propagation mechanism of technology shocks. A technology shock is associated with a larger capital stock, which enables a larger increase in consumption (but of an order of magnitude smaller than capital in terms of second order contribution, as a precautionary reaction) and a smaller increase in labor (due to the second order downward correction) than the linear model would predict. In the case of a negative technology shock (not pictured), the first order components would simply be their mirror images with opposite sign. The second order contributions, however, would remain unchanged due to the symmetry of the quadratic function. In combination, the second order approximation can thus capture time invariant asymmetries in the impulse responses.³⁶

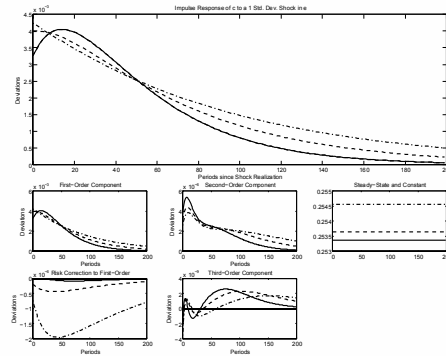
The precautionary component can likewise be seen in the upward correction of the steady states in the rightmost panels. In the stochastic steady state, agents face uncertainty regarding future shocks and accumulate a precautionary stock of capital through increased labor efforts and disburse this as increased consumption when shocks fail to manifest themselves. The lower left panel contains the contributions from $y_{\sigma^2,i}$ the second order (in σ) time varying correction for risk, this demonstrates an initial wealth effect that complicates the response of consumption relative to a nonstochastic environment.³⁷ While capital and labor decrease, consumption displays an initial small decrease, then recovery, before facing a large downward adjustment; this initial recovery is a wealth effect that corrects consumption upwards back toward zero (see figure 3.2b, when risk aversion is increased, the precautionary effect dwarfs this wealth effect). Nonlinear impulse responses are not scale invariant, as noted also by Fernández-Villaverde et al. (2011b): while the first order component scales linearly with the magnitude of the shock, the second order order component scales quadratically. As shocks become larger, a linear approximation would generally not suffice to characterize the dynamics of the model. This is precisely the effect of higher order terms: as the magnitude of the shock increases, these higher order terms begin to contribute more significantly to the total impulse, correcting the responses for the greater departure from the steady state. For this model, however, one would need to consider shocks of unreasonable magnitude to generate any notable effects from the higher order terms on the total impulse, reinforcing the conventional wisdom that this model is nearly linear.

³⁶Time varying asymmetries would be captured by $y_{\sigma^2,j,i}$, require a fourth order approximation as the term $y_{\sigma,j,i}$ from the third order approximation is zero, see section 3.3.3.

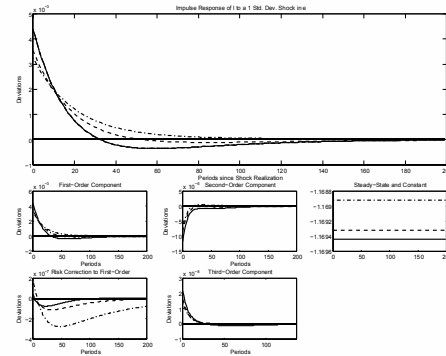
³⁷Fernández-Villaverde and Rubio-Ramírez (2010b) discusses the nonlinear impact of shocks in the production function and similar wealth effects.



(a) Capital



(b) Consumption



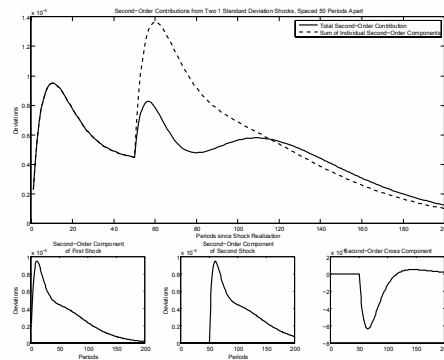
(c) Labor

Fig. 3.2 Impulse Responses to a Technology Shock, Model of Section 3.5.2
Solid: $\gamma = 2$, Dashed $\gamma = 5$, Dash-Dotted $\gamma = 10$

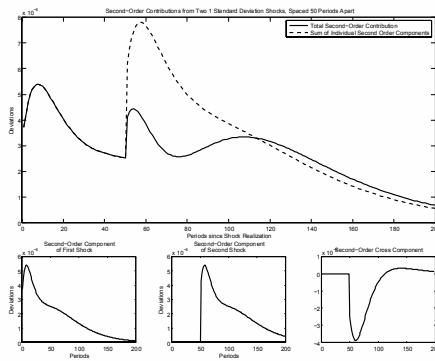
In figure 3.2, the impulse responses to a technology shock with different values (2, 5, and 10) of the CRRA parameter γ are overlayed. Note that for all three values of γ , the first order components dominate. While changes in γ do change the periodicity of the harmonic distortion as well as the shape and sign of some second and third order components, the constant and time varying corrections for risk display a significant change in magnitude. As γ is increased, the stochastic steady state is associated with higher constant precautionary stocks of capital and the time varying precautionary reduction in the response of consumption eclipses the wealth effect. Though not very large, the second order kernel is highlighted by the experiment, with both the second order contributions of capital and labor increasing minimally relative to their responses with $\gamma = 2$ and that of consumption decreasing relatively initially. At values above 20 (not pictured), the time varying corrections for risk begin to contribute noticeably to the total impulse, whereas shocks several orders of magnitude larger than a standard deviation are needed to propel the nonlinear kernels to significance.

Figure 3.3 highlights a central component of higher order impulse responses: the breakdown of superposition or history dependence of the transfer function. The nonlinear impulse response to two shocks at different points in time is not equal to the sum of the individual responses, even after having corrected the individual responses for the higher order. The panels in the figure depict the second order contributions to the impulse responses of capital, consumption, and labor to two positive, one standard deviation technology shocks, spaced 50 periods apart. The dashed line in the top of figure simply adds the individual second order components from each shock together (i.e., presents the total second order component if superposition were to hold), whereas the solid line additionally contains the second order cross component (i.e., presents the true total second order component). Demonstrating this breakdown of superposition, the cross component overwhelms the individual components shortly after the second shocks hits and the second order contributions to the responses of capital (upper panel) and consumption (middle panel) fail to match the peak response from a single shock, despite the lingering contribution from the initial shock in the same direction. Although the mitigation is much less pronounced for labor (lower panel), the difference from the sum of individual contributions is nonetheless noticeable and prolonged. In a nonlinear environment, there is no single measure for an impulse response;³⁸ in starting from the stochastic steady state, however, we remove any deterministic trends (as would be present, e.g, when starting from the ergodic mean, see footnote 34) in our impulse response measure at each order of approximation.

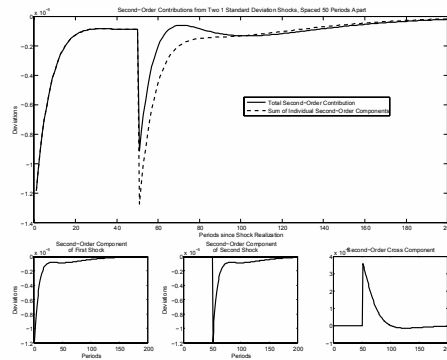
³⁸See, e.g., Koop et al. (1996), Potter (2000), and Gourieroux and Jasiak (2005).



(a) Capital



(b) Consumption



(c) Labor

Fig. 3.3 Second-Order Contributions to Impulse Responses to a Technology Shock, Model of Section 3.5.2

3.6 Accuracy

In this section, we explore the accuracy of our solution method using Euler equation errors.³⁹ We validate the accuracy of our solution method and we add an Euler equation error method for assessing the accuracy of an impulse response to address our infinite-dimensional state space.

We examine our method using the model of Aruoba et al. (2006), examined in section 3.5.2. From Judd (1992), the idea of the Euler equation accuracy test in the neoclassical growth model is to find a unit free measure that expresses the one period optimization error in relation to current consumption. Accordingly, (3.60) can be rearranged to deliver the Euler equation error function as⁴⁰

$$EE() = 1 - \frac{1}{C_t} \left(\frac{\beta E_t \left[\frac{(C_{t+1}^\theta (1-L_{t+1})^{1-\theta})^{1-\gamma}}{C_{t+1}} (\alpha e^{Z_{t+1}} K_t^{\alpha-1} L_{t+1}^{1-\alpha} + 1 - \delta) \right]}{(1-L_t)^{(1-\theta)(1-\gamma)}} \right)^{\frac{1}{\theta(1-\gamma)-1}} \quad (3.66)$$

Deviations in (3.66) from zero are interpreted by Judd (1992) and many others as the relative optimization error that results from using a particular approximation. Expressed in absolute value and in base 10 logarithms, an error of -1 implies a one dollar error for every ten dollars spent and an error of -6 implies a one dollar error for every million dollars spent.

The arguments of $EE()$ depend on the state space postulated. Standard state space methods would choose $EE(K_{t-1}, Z_t)$ or $EE(K_{t-1}, Z_{t-1}, \varepsilon_{Z,t})$. Our nonlinear moving average policy function requires $EE(\varepsilon_{Z,t}, \varepsilon_{Z,t-1}, \dots)$, rendering the Euler equation error infinite dimensional. In line with our presentation of impulse responses, we examine the following set of Euler equation error functions, holding all but one shock constant and moving back in time from t , assessing the one-step optimizing error of the impulse responses.

$$EE_t = EE(\varepsilon_{Z,t}, 0, 0, \dots), EE_{t-1} = EE(0, \varepsilon_{Z,t-1}, 0, \dots), \dots \quad (3.67)$$

We examine a range of shock values for $\varepsilon_{Z,t-j}$ that covers 10 standard deviations in either direction. This is perhaps excessive given the assumption of normality, but enables us to cover the same range from a single shock that Aruoba et al. (2006) examine for the technology process. Figure 3.4 plots EE_t for first through third order approximations in logs, see section 3.5.2, and in the variables' original level specification. The

³⁹See, e.g., Judd (1992), Judd and Guu (1997), and Judd (1998)

⁴⁰Cf. Aruoba et al. (2006, p. 2499).

first observation is that moving to a higher order uniformly increases accuracy—this result is reassuring, but not a given. As Lombardo (2010, p. 22) remarks, although within the radius of convergence the error in approximation goes to zero as the order of approximation becomes infinite, this does not necessarily happen monotonically. If we restrict our attention to three standard deviation shocks (± 0.021), the second order log approximation make mistakes no greater than one dollar for everyone ten million spent and the third order level and log approximations no greater than one dollar for everyone one-hundred million and one billion spent respectively, hardly an unreasonable error. Of independent interest is the result that the first order approximation in logs is uniformly superior to the first order approximation in levels, in contrast to Aruoba et al. (2006). As their mapping was from capital to errors and ours from shocks to errors, the preferred approximation appears to depend on the dimension under study.

In figure 3.5, plots of EE_{t-j} for $j = 0, 1, \dots, 100$ for the first order approximations in both levels and logs are provided. Comparing these two figures—let alone incorporating the associated results for the second and third order (not pictured)—is difficult at best. Thus, to facilitate comparison of the different approximations across the different horizons, two measures that reduce to two dimensions will be examined, namely maximal and average Euler equation errors.

First, we plot the maximal Euler equation errors over a span of 100 periods in figure 3.6. I.e.,

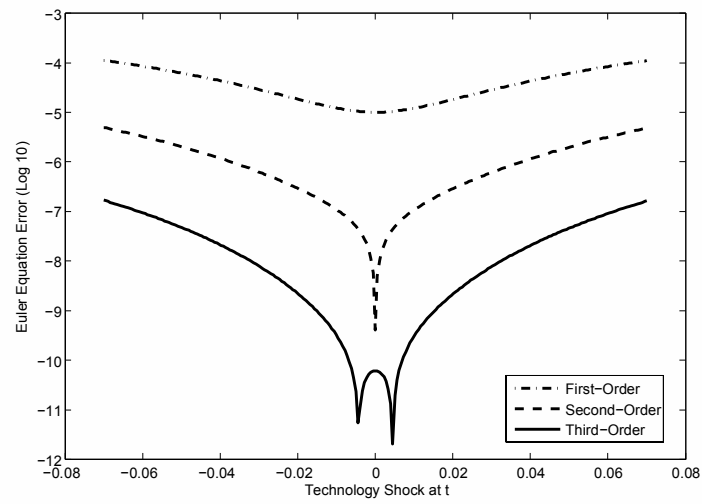
$$\max_{-10\sigma_Z < \varepsilon_{Z,t-j} < 10\sigma_Z} (EE_{t-j}), \text{ for } j = 0, 1, \dots, 100 \quad (3.68)$$

where σ_Z is the standard deviation of the technology shock, see table 3.1. The figure tends to reinforce the results from examining only shocks in period t : for both the level and log approximations, moving to a higher order uniformly improves the quality of approximation and, at all three orders, moving from a level to a log specification likewise improves the accuracy of the approximation uniformly according to this metric.

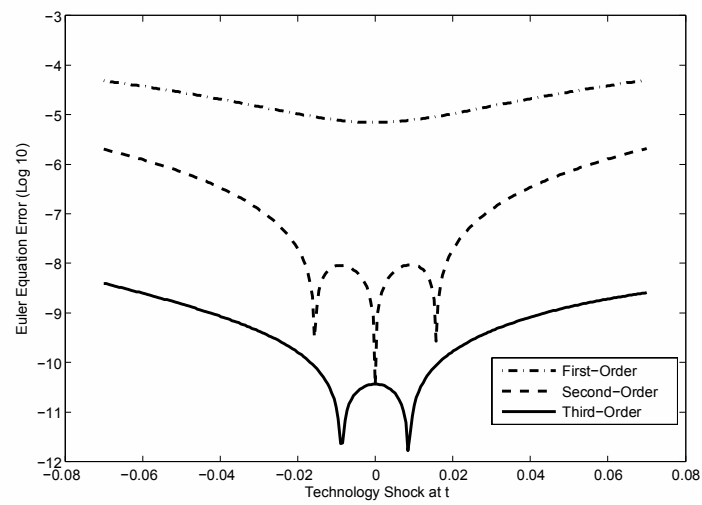
In our final measure, we graph average Euler equation errors over 100 periods in figure 3.7. Whereas state space measures require the ergodic distribution of endogenous state variables, our measure is relatively easy to calculate, as we merely need to integrate with respect to the known distribution (in this case normal) of the shocks

$$\int EE_{t-j} dF_{\varepsilon_{Z,t-j}}, \text{ for } j = 0, 1, \dots, 100 \quad (3.69)$$

Weighting the regions of shock realizations most likely to be encountered as defined by the distribution of shocks, we are not forced to make a choice regarding the range of shock values to consider. Again, we note the uniform improvement with higher order and the improvement in the approximation by switching to logs. The average error us-

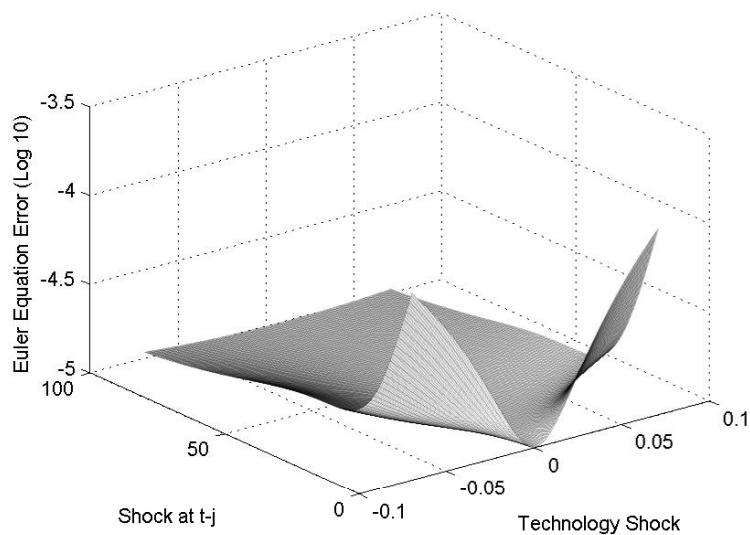


(a) In Levels

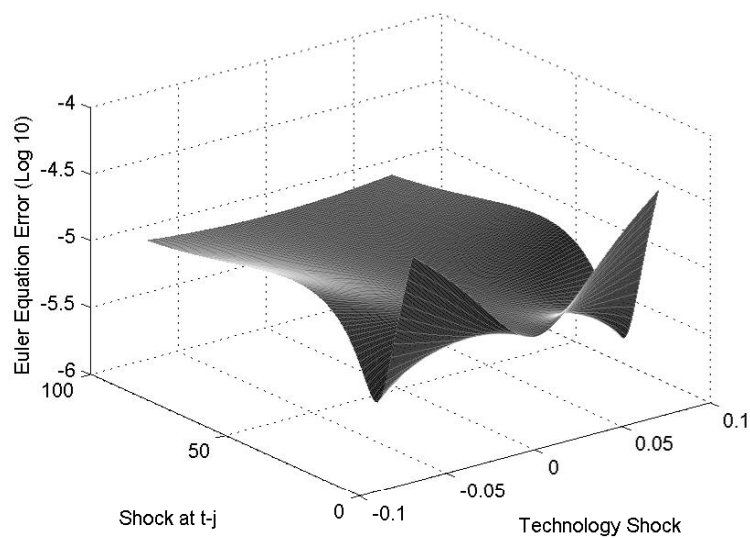


(b) In Logs

Fig. 3.4 Euler Equation Errors, Shock at Time t , Aruoba et al.'s (2006) Baseline Case

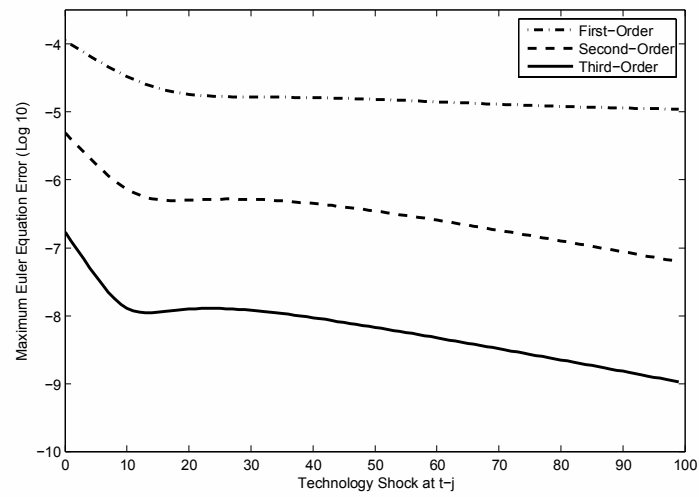


(a) In Levels

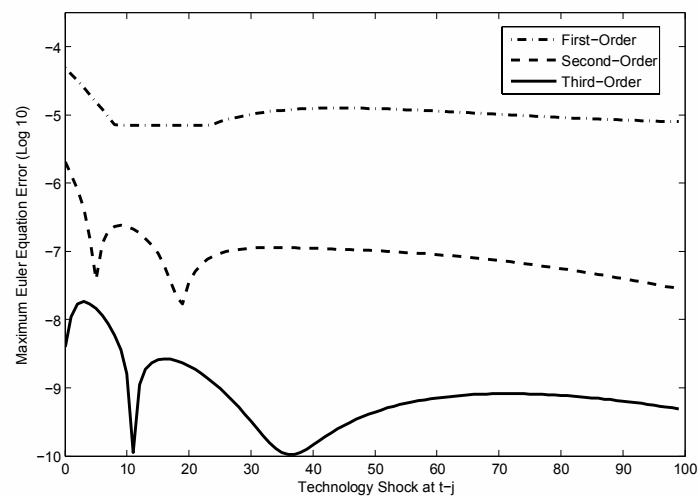


(b) In Logs

Fig. 3.5 Euler Equation Errors, First Order Approximation, Aruoba et al.'s (2006) Baseline Case



(a) In Levels



(b) In Logs

Fig. 3.6 Maximum Euler Equation Errors, Aruoba et al.'s (2006) Baseline Case

ing a first order in level approximation is around one dollar for every ten thousand spent regardless of horizon. The second order approximations show an improvement as the horizon increases, whereas the third order approximations display little improvement across horizons. The third order approximation in both levels and logs are associated with an average error of less than one dollar for every ten billion spent regardless of horizon.

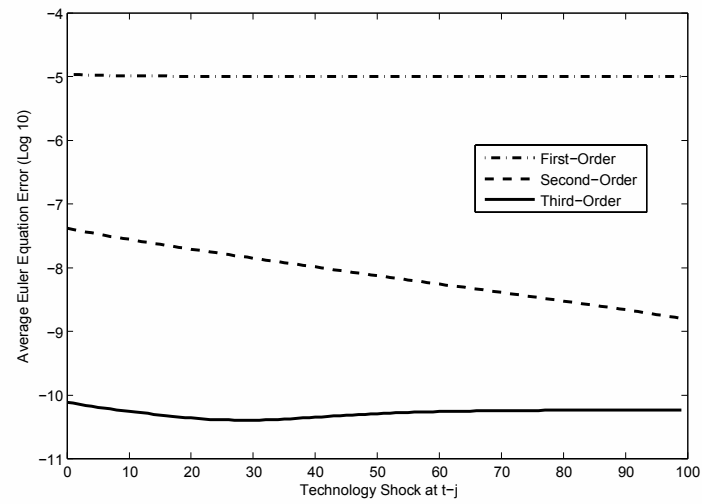
We conclude that the nonlinear moving average policy function can provide competitive approximations of the mapping from shocks to endogenous variables. As was the case with Aruoba et al. (2006), however, the perturbation methods here deteriorate (not reported) in their extreme parameterization. Our method remains a local method and is subject to all the limitations and reservations that face such methods.

3.7 Conclusion

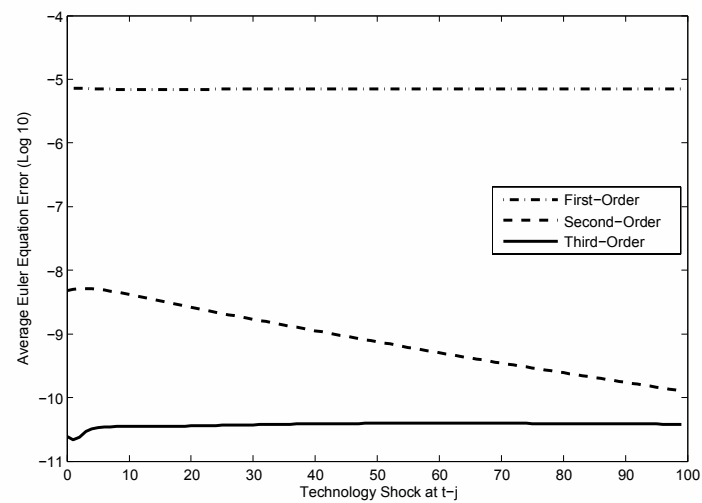
We have introduced a nonlinear infinite moving average as an alternative to the standard state space policy function to the analysis of nonlinear DSGE models. We have derived a perturbation approximation of this policy function, providing explicit derivations up to third order in the form of a Volterra expansion. This direct mapping of the history of shocks into endogenous variables enables familiar impulse response analysis techniques in a nonlinear environment and provides a convenient decomposition of the mapping into approximation and uncertainty orders. We confirm that this approach provides a degree of accuracy comparable to state space methods by introducing Euler equation error methods for this infinite dimensional mapping.

Although there are a number of DSGE models and applications, for example, welfare analysis, asset pricing and stochastic volatility for which the importance of nonlinear components and uncertainty in the policy function has been proved, the nonlinear components we analyzed in the baseline neoclassical growth model are quantitatively unimportant. This is not surprising as the model is known to be nearly linear. Qualitatively however, the nonlinear contributions to the the mapping from shocks to endogenous variables are economically interpretable, translating, e.g., into precautionary behavior and wealth effects. Likewise, non economically interpretable artifacts of the nonlinear method, such as harmonic distortion are documented as well.

The potential for explosive behavior in the simulation of state space perturbations has lead to the adaptation of ‘pruning’ algorithms, see Kim et al. (2008), that appear ad-hoc relative to the perturbation solution itself. With our method, however, the stability from the first order solution is passed on to all higher order recursions. This feature



(a) In Levels



(b) In Logs

Fig. 3.7 Average Euler Equation Errors, Aruoba et al.'s (2006) Baseline Case

of the nonlinear kernels in our moving average solution is consistent with the Volterra operator acting upon the history of shocks being bounded, alluding to the existence of an endogenous ‘pruning’ algorithm derived from inverting our moving average, which we study in ongoing research, see Lan and Meyer-Gohde (2013a).

The nonlinear perturbation DSGE literature is still in development and our method provides a different perspective by mapping directly from the history of shocks. Standard state space DSGE perturbation methods provide insight into the nonlinear mapping between endogenous variables through time. Yet when the researcher’s interest lies in examining the nonlinear mapping from exogenous shocks to endogenous variables, our method has additional insight to offer.

3.8 Acknowledgements

We are grateful to Michael Burda, Giovanni Lombardo, Lutz Weinke, Peter Zadrozny, two anonymous referees, and the editor, Paul Klein, as well as participants of the CEF 2012 and of research seminars at the Bundesbank and HU Berlin for useful comments, suggestions, and discussions. This research was supported by the DFG through the SFB 649 “Economic Risk”. Any and all errors are entirely our own.

Chapter 4

Decomposing Risk in Dynamic Stochastic General Equilibrium

Hong Lan[†]

Alexander Meyer-Gohde[§]

Abstract

We analyze the theoretical moments of a nonlinear approximation to real business cycle model with stochastic volatility and recursive preferences. We find that the conditional heteroskedasticity of stochastic volatility operationalizes a time-varying risk adjustment channel that induces variability in conditional asset pricing measures and assigns a substantial portion of the variance of macroeconomic variables to variations in precautionary behavior, both while leaving its ability to match key macroeconomic and asset pricing facts untouched. We calculate the theoretical moments directly and decomposes these moments into contributions from shifts in the distribution of future shocks (i.e., risk) and from realized shocks and differing orders of approximation, enabling us to identify the common channel through which stochastic volatility in isolation operates and through which conditional asset pricing measures vary over time. Under frictional investment and varying capital utilization, output drops in response to an increase in risk, but the contributions to the variance of macroeconomic variables from risk becomes negligible.

JEL classification: C63, E32, G12

[†]Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 1466; Email: lanhong@cms.hu-berlin.de

[§]Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 5720; Fax: +49-30-2093 5696; E-Mail: alexander.meyer-gohde@wiwi.hu-berlin.de

Keywords: Recursive preferences; stochastic volatility; asset pricing; DSGE; moment calculation

4.1 Introduction

Assessing the statistical and structural implications of nonlinear DSGE models with recursive preferences and stochastic volatility for asset pricing and business cycle dynamics is an unfinished task in macroeconomics. We derive the theoretical moments of nonlinear moving average approximations to the model and decompose these moments into contributions from the individual orders of nonlinearity in realized shocks (amplification effects) and from the moments of future shocks (risk adjustment effects). With this decomposition, we find that stochastic volatility activates a time-varying risk adjustment channel in macroeconomic variables accounting for a substantial amount of total variation. We identify this conditional heteroskedastic mechanism as the sole driving force of the conditional asset pricing measures under study. This enables us to tell the story of a varying pattern of risk in the economy eliciting changes in households' precautionary responses as priced by measures such as the conditional market price of risk. We find, however, that stochastic volatility contributes to the model's ability to match asset pricing facts only by increasing the overall volatility of macro variables—taken as given exogenously in endowment settings¹ that reach the opposite conclusion—and that frictional investment and variable capital utilization allow the model to predict a drop in output in response to an increase in risk (positive volatility shock) at the cost of making the importance of this risk channel to the variability of macro variables moot.

While there is growing interest in stochastic volatility and Epstein and Zin's (1989) recursive preferences² in recent literature, there is little work that studies the joint effect of these two elements for both asset pricing and business cycle dynamics.³ Andreasen

¹See, e.g., Bansal and Yaron (2004)

²See also Kreps and Porteus (1978) and Weil (1990). Backus et al. (2005) offers a recent review of these and related preferences.

³Bloom (2009) studies the impact of stochastic volatility at the firm level and documents a short drop followed by an overshooting in aggregate economic activity following a volatility shock. Justiniano and Primiceri (2008) add stochastic volatility to a linearized New Keynesian model to study the documented reduction in volatility of U.S. economy since the early 1980's (See Blanchard and Simon (2001) and Stock and Watson (2003), as well as Sims and Zha (2006) for a review.). Fernández-Villaverde et al. (2011a) and Born and Pfeifer (2013) use New Keynesian models to study the effect of changes in the volatility of policy variables on the aggregate economy. Tallarini (2000) among others, note recursive preferences can contribute to resolving the longstanding asset pricing puzzles (equity premium and risk free rate) documented in Mehra and Prescott (1985) and Weil (1989) without compromising the

(2012b), focusing on the different specifications of the conditional heteroskedasticity and the consequential difference in the quantitative performance of a New Keynesian model, takes a brief look at the implications of the model on both sides. Bidder and Smith (2012), taking a model uncertainty perspective à la Hansen and Sargent (2007), study fluctuations in the worst-case distribution as sources for business cycles in a model with stochastic volatility and recursive preferences. We differ from both their works in our aim to analyze the propagation mechanism of stochastic volatility implemented as a volatility shock in a production model, and we examine the role of stochastic volatility in attaining the Hansen-Jagannathan bounds (See Hansen and Jagannathan (1991)) without compromising the fit vis-à-vis the macroeconomy to complement the empirical evaluation of the model regarding replicating asset pricing regularities.

We solve the model using the nonlinear moving average perturbation derived in Lan and Meyer-Gohde (2013d), which takes the infinite sequence of realized shocks, past to present, as its state variable basis and adjusts the deterministic policy function for the effect of future shocks by scaling their distribution with the perturbation parameter. Following Caldara et al.'s (2012) assessment of the accuracy of third order perturbations in a business cycle model with recursive preferences and stochastic volatility and as it is the minimum order needed to capture the time-varying shifts in risk premium as noted in Andreasen (2012b, p. 300) and van Binsbergen et al. (2012, p. 638), we approximate the policy function to third order. The nonlinear moving average policy function and its third order approximation can be decomposed straightforwardly into the order of the amplification effects (the impact of the realized shocks) and risk adjustment (the anticipation effect of future shocks). We find, in the analysis of the impulse responses of both macroeconomic and asset pricing variables, a volatility shock by itself propagates solely through the time-varying risk adjustment channel. For conditional asset pricing measures such as the expected risk premium, volatility shocks and productivity growth shocks propagate individually through the time-varying risk adjustment channel only. Moreover, the effect of stochastic volatility shocks on the expected risk premium is several orders of magnitude larger than that of productivity growth shocks, highlighting again the importance of this time variation in the dispersion of probability measures used to form expectations for conditional asset pricing.

Using the third order nonlinear moving approximation, we derive theoretical moments that are in general not available in the nonlinear DSGE models.⁴ In a similar

model's ability of replicating macroeconomic dynamics; and Rudebusch and Swanson (2012) and van Binsbergen et al. (2012) use a model with recursive preferences to study the dynamics of the yield curve.

⁴The nonlinear moving average approximation, as its policy function directly maps exogenous

vein to our nonlinear moving average, Andreasen et al. (2013) compute theoretical moments using a pruned state space perturbation,⁶ since after pruning, the unknown higher moments are nonlinear functions of the known moments of lower order approximations. However, we are able to further derive a decomposition of the theoretical variance that neatly dissects the individual contributions of amplification and risk adjustment effects to the total variance of the model. With this variance decomposition, we find that adding stochastic volatility changes the composition of the variance of the macroeconomic variables. In the presence of stochastic volatility, more variation is generated in the time-varying risk adjustment channel. As for macroeconomic variables, movements in the risk adjustment channel can be explained by the household's precautionary motive. This finding implies households aware of shifts in the distributions of future shocks will adjust their precautionary behavior commensurately.

The paper is organized as follows. The competitive real business cycle model with recursive preferences and stochastic volatility is derived in section 5.2. In section 4.4, we present the nonlinear moving average perturbation solution to the model. The calibrations are introduced in section 4.5. We then derive the theoretical moments in section 4.6 and apply our method to analyze the model in section 5.4. In section 4.8, we extend the baseline model to frictional investment and variable capital utilization. Section 5.6 concludes.

4.2 Stochastic Volatility

4.2.1 Related Literature

As documented in Blanchard and Simon (2001), Stock and Watson (2003), Sims and Zha (2006), Fernández-Villaverde and Rubio-Ramírez (2010a) and many others, the volatility of employment growth, consumption growth and output of the U.S. economy from 1984 to 2007 has evidently declined by one third comparing to their values during the 1970s and early 1980s. Nominal volatilities also have declined by more than half. This period of volatility reduction in aggregate time series, often labeled as the Great Moderation, motivates the study of its causes. The literature thus far offers three main

shocks into the endogenous variables, only needs the moments of the exogenous shocks when computing the theoretical moments. We implement our approach numerically by providing an add-on for the popular Dynare package.⁵ A state space perturbation policy function, by contrast, maps the endogenous variables into themselves and resulting in an infinite regression in theoretical moments requiring higher moments than moments being computed.

⁶See Lan and Meyer-Gohde (2013b) for an overview of pruning and its relation to our nonlinear moving average.

ways of modeling, and therefore analyzing this volatility shift: i) stochastic volatility, i.e., model the volatility of the exogenous processes under investigation as an autoregressive process, or ii) a GARCH process, or iii) the volatility switches between two (or more) states, i.e., Markov regime switching models. As pointed out by Fernández-Villaverde and Rubio-Ramírez (2010a, p. 10), stochastic volatility can capture many important features of the empirical volatility shift and differentiates the special effect of volatility from others, this approach has been adopted in many studies.

By incorporating stochastic volatility, Fernández-Villaverde and Rubio-Ramírez (2007) show variations in the volatility of investment-specific technological shock and preferences shock explain most of variations in the volatility of in output and hours worked. Justiniano and Primiceri (2008) estimate a DSGE model with stochastic volatility and conclude the decline in the volatility of output, hours worked and consumption is largely owing to a change in the variance of the investment-specific technological shock, e.g., a change in the variance of the investment shock explains on average 30% of variability in output growth since mid-1980s. On the other hand, Bidder and Smith (2013) investigate the implication of stochastic volatility on asset pricing and find that, their endowment model economy generates a much higher unconditional market price of risk with stochastic volatility in the exogenous consumption growth process than without. Meanwhile, the presence of stochastic volatility does not lead to a noticeable change in risk free rate, and thus improves the model's ability in attaining the Hansen-Jagannathan bound. Bansal and Yaron (2004) also find, in the presence of stochastic volatility in the exogenous consumption growth, the maximal Sharpe ratio, i.e., the lower bound of the market price of risk, is about three times larger than its value in the absence of stochastic volatility.

All these findings motivates the study of whether stochastic volatility is a driver of business cycle fluctuations. As noted in Born and Pfeifer (2013), changes in aggregate uncertainty can potentially induce changes in economic activities through i) the precautionary motive as household tends to save more to ensure itself against the increased future risk, ii) the (inverse) Hartman-Abel effect which is, in essence, firm's precautionary reaction in response to the increased future risk, and iii) real option effect at work. These three transmission mechanisms of volatility change are however, partial equilibrium effects. In a general equilibrium model where prices can adjust to accommodate changes in uncertainty, the effect of volatility change on economic activities could differ from that in a partial equilibrium model, both qualitatively and quantitatively.

Using a partial equilibrium model, Bloom (2009) show a positive volatility shock

causes a drop in output and employment, both by about 1%. Bloom et al. (2012) report a drop in output by just over 3% when general equilibrium effects are shut off. On the other hand, the effect of stochastic volatility appears, but not without exception, smaller in size when general equilibrium effects are taken into account. Fernández-Villaverde et al. (2011b) show a volatility shock to the real interest rate leads to a drop in output, consumption, investment and hours by 0.2%, 0.5%, 2% and 0.001% respectively.⁷ Fernández-Villaverde et al. (2011a) find the previous four aggregates fall by about 0.15%, 0.02%, 0.6% and 0.15% respectively in response to an increase in the uncertainty of fiscal policy. Born and Pfeifer (2013) also study the uncertainty of fiscal policy and its effect. They also find, with the baseline parameterization of their model, an increase in uncertainty causes a contraction on economic activities—the four aggregates fall by 0.045%, 0.03%, 0.1% and 0.04% respectively. Basu and Bundick (2012) investigate the effect of stochastic volatility in both technology and preference shock process. A volatility shock to either of the two processes leads to a drop in output, consumption, investment and hours, but at different quantitative level. The four aggregates fall by about 0.04%, 0.06%, 0.01% and 0.06% respectively in response to a volatility shock to technology, and by about 0.17%, 0.16%, 0.2% and 0.21% in response to a volatility shock to preference. While the first order impact effect of aggregate uncertainty is not pictured and explicitly reported, Bachmann and Bayer (2013) show the contribution from aggregate uncertainty to the volatility of output, consumption, investment and hours is negligible. Still in a general equilibrium framework, Bloom et al. (2012) and Bidder and Smith (2013) both find however, stochastic volatility can have large effect on economic activities. A volatility shock causes a fall in output, consumption, invest and hours by about 3%, 1%, 20% and 7% in Bloom et al. (2012), and by about 2%, 1.5%, 2.5% and 1% in the worse case model of Bidder and Smith (2013).

To summarize, the studies cited above tend to agree an increase in volatility leads to a recession, yet differ in their views on the size of such a recession. Moreover, Bloom (2009) and Bloom et al. (2012) find the recession due to the increase in uncertainty lasts for only 6 and 12 months respectively, as opposed to a more prolonged recession reported by the other studies. Additionally, there is few studies investigating the potential effect of stochastic volatility on the market price of risk in a production economy. It worth noting that the contribution from stochastic volatility to asset pricing in an endowment economy does not necessarily carry over to a production economy, as such contribution is not entirely independent from the reduced form, empirical specification

⁷This is the case for Argentina, see Fernández-Villaverde et al. (2011b, p. 2550).

of consumption growth.

4.2.2 General Operation within DSGE Models

One general way⁸ to introduce stochastic volatility is to replace a homoskedastic shock ω_t with $e^{\zeta_t} \omega_t$, where ζ_t is a mean zero stochastic variable. The exponential function ensure that e^{ζ_t} is always positive, enabling the interpretation of the product of e^{ζ_t} and homoskedastic standard deviation of ω_t as the shock's conditional standard deviation. As we will be concerned with local approximations in this study, it will be useful to have a Taylor series of a conditionally heteroskedastic shock $e^{\zeta_t} \omega_t$ as we approximate our equilibrium system to a given order.

Lemma 4.2.1. *The Taylor expansion of*

$$e^{\zeta_t} \omega_t \quad (4.1)$$

around the point $\zeta_t = \omega_t = 0$ is given by

$$e^{\zeta_t} \omega_t = \left(\sum_{i=0}^{\infty} \frac{1}{i!} \zeta_t^i \right) \omega_t \quad (4.2)$$

Proof. See the appendices. \square

To assess the general equilibrium effects of stochastic volatility, consider the following general model

$$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + H e^{\zeta_t} \omega_t] \quad (4.3)$$

where y_t is the vector of the endogenous variables, and ε_t the vector of normally distributed exogenous shocks apart from the stochastic volatility shock under consideration, H a constant vector, ω_t is a normally distributed exogenous shock subjected to a stochastic volatility process ζ_t ,⁹ itself given by

$$\zeta_t = \rho \zeta_{t-1} + \tau \eta_t \quad (4.4)$$

where $|\rho| < 1$ is the autocorrelation of the process, η_t its standard normal innovation with $\tau > 0$ scaling these innovations to enable non-unity standard deviations.

The solution to (4.3) is a time-invariant function y , taking as its state variable basis, the shocks and states induced by stochastic volatility $z_t^\omega \equiv [\zeta_{t-1} \quad \varepsilon_t^{\omega'}]'$, where $\varepsilon_t^\omega \equiv [\omega_t \quad \eta_t]'$, as well as the remaining shocks and states, $z_t \equiv [y'_{t-1} \quad \varepsilon_t']'$. Solutions

⁸See, e.g., Fernández-Villaverde et al. (2011b)

⁹Our shock subject to stochastic volatility enters the model linearly through $H e^{\zeta_t} \omega_t$. The vector H could, for example, contain zeros everywhere but for the row associated with, say, an autoregressive process for technology; in this row, the entry would be the homoskedastic standard deviation of technology shocks.

are indexed by the perturbation parameter $\sigma \in [0, 1]$ scaling the distribution of future shocks

$$y_t = g(\sigma, z_t, z_t^\omega) \quad (4.5)$$

$$y_{t+1} = g(\sigma, [y'_t \quad \sigma \varepsilon'_{t+1}]', [\varsigma_t \quad \sigma \varepsilon'_{t+1}]') \quad (4.6)$$

The role of the parameter $\sigma \in [0, 1]$ can be seen in the policy function at $t + 1$, where it premultiplies shocks dated $t + 1$. That is, from the time t perspective of the conditional expectations operator in (4.3), shocks dated $t + 1$ are unknown and are the source of risk in the model which is scaled by σ . When $\sigma = 0$, the model is deterministic and the deterministic steady state is a fix point of the mappings in (4.5) when all shock realizations are equal to their mean (i.e., zero) values.

We are now in a position to provide the third order Taylor approximation of (4.5) about the deterministic steady state.

Proposition 4.2.2. *The recursive solution of (4.3) expanded out to third order at the deterministic steady state is given by*

$$\begin{aligned} y_t \approx & \frac{1}{2} g_{\sigma^2} + \frac{1}{2} g_{\sigma^2 z} z_t + \frac{1}{2} g_{\sigma^2 z^\omega} z_t^\omega \\ & + (g_{\varsigma \omega} \varsigma_{t-1} + g_{\eta \omega} \eta_t) \omega_t \\ & + \frac{1}{2} (g_{\varsigma \omega z} \varsigma_{t-1} + g_{\eta \omega z} \eta_t) \omega_t z_t \\ & + \frac{1}{2} (g_{\varsigma^2 \omega} \varsigma_{t-1}^2 + 2g_{\varsigma \eta \omega} \varsigma_{t-1} \eta_t + g_{\eta^2 \omega} \eta_t^2) \omega_t \\ & + \text{Terms independent of stochastic volatility} \end{aligned} \quad (4.7)$$

Proof. See the appendices. □

Notice, importantly, that only derivatives with respect to the risk scaling parameter σ provide a channel for stochastic volatility to interact with variables outside of the stochastic volatility and the shock it impacts. The remaining terms, i.e., those that do not involve derivatives with respect to the risk scaling parameter σ , capture the direct effect of a change in volatility on shocks drawn from the distribution subjected to the change. We split the third order approximation of the recursive solution of (4.3) in proposition 4.2.2 into these two components.

First, the component that captures the effect of changes in the dispersion of the distribution of shocks on the magnitude of shocks realized from this distribution.

Definition 4.2.3. *Amplification Component At third order, the amplification component*

of y_t is

$$\begin{aligned} y_t^{amplification} &= (g_{\varsigma\omega}\varsigma_{t-1} + g_{\eta\omega}\eta_t) \omega_t \\ &\quad + \frac{1}{2} (g_{\varsigma\omega z}\varsigma_{t-1} + g_{\eta\omega z}\eta_t) \omega_t z_t \\ &\quad + \frac{1}{2} (g_{\varsigma^2\omega}\varsigma_{t-1}^2 + 2g_{\varsigma\eta\omega}\varsigma_{t-1}\eta_t + g_{\eta^2\omega}\eta_t^2) \omega_t \end{aligned} \quad (4.8)$$

In essence, an increase in the dispersion of this distribution serves to magnify the realizations of shocks from the undispersed distribution, hence our label, “amplification.”

Second, the component that captures the effect of changes in the dispersion of the distribution of shocks on the evaluation of expectations

Definition 4.2.4. *Risk Component At third order, the risk component of y_t is*

$$y_t^{risk} = \frac{1}{2}g_{\sigma^2} + \frac{1}{2}g_{\sigma^2 z}z_t + \frac{1}{2}g_{\sigma^2 z^w}z_t^w \quad (4.9)$$

In essence, an increase in the dispersion of this distribution increase the risk or measurable uncertainty regarding future stochastic variables, hence our label, “risk.”

The volatility shock η_t can only affect the conditional distribution of future shocks only if the stochastic process ς_t is persistent, as we summarize in the following

Corollary 4.2.5. *Risk Component and Persistence At third order, the risk component of y_t , (4.9), is nonzero if and only if ς_t is persistent, that is if $\rho \neq 0$.*

Proof. See the appendices. □

We have defined the volatility process in (4.1) such that changes in the volatility occur simultaneous with realizations of shocks, that is both ς_t and ω_t enter (4.1) with a subscript t . Volatility shocks will only affect risk or measurable uncertainty regarding future shocks if ς_t is serially correlated, such that an innovation to the volatility process also affects future volatility.

4.3 The Model

In this section, we derive a stochastic neoclassical growth model with the recursive preferences and stochastic volatility. We will follow Tallarini (2000) closely so that our model coincides with his in the special case of constant (i.e., non stochastic) volatility. Preferences are recursive in an exponential certainty equivalent with the period utility

function logarithmic in consumption and leisure. Production is neoclassic using time-to-build capital and labor, whose productivity grows as a random walk with drift and innovations subject to stochastically varying volatility.

The economy is populated by an infinitely lived household seeking to maximize its expected discounted lifetime utility given by the recursive preferences

$$V_t = \ln C_t + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} V_{t+1} \right) \right] \right) \quad (4.10)$$

where C_t is consumption, N_t labor, $\beta \in (0, 1)$ the discount factor and

$$\gamma \equiv 2 \frac{(1 - \beta)(1 - \chi)}{1 + \psi} \quad (4.11)$$

indexes the deviation with respect to the expected utility. χ denotes the coefficient of relative risk aversion (CRRA) for atemporal wealth gambles¹⁰ and $\psi > 0$ controls labor supply. With χ equal to the elasticity of intertemporal substitution (EIS) which is equal to one here, (4.10) collapses to the expected utility. The household optimizes over consumption and labor supply subject to

$$C_t + K_t = W_t N_t + r_t^K K_{t-1} + (1 - \delta) K_{t-1} \quad (4.12)$$

where K_t is capital stock accumulated today for productive purpose tomorrow, W_t real wage, r_t^K the capital rental rate and $\delta \in [0, 1]$ the depreciation rate. Investment is the difference between the current capital stock and the capital stock in the previous period after depreciation

$$I_t = K_t - (1 - \delta) K_{t-1} \quad (4.13)$$

We assume a perfectly competitive production side of the economy, where output is produced using the labor augmented Cobb-Douglas technology $Y_t = K_{t-1}^\alpha (e^{Z_t} N_t)^{1-\alpha}$. Z_t is a stochastic productivity process and $\alpha \in [0, 1]$ the capital share. Productivity is assumed to be a random walk with drift, incorporating long-run risk into the model¹¹

$$a_t \equiv Z_t - Z_{t-1} = \bar{a} + \bar{\sigma}_z e^{\sigma_{z,t}} \varepsilon_{z,t}, \quad \varepsilon_{z,t} \sim \mathcal{N}(0, 1) \quad (4.14)$$

with $\varepsilon_{z,t}$ the innovation to Z_t . $\bar{\sigma}_z e^{\sigma_{z,t}}$ can be interpreted as the standard deviation of the productivity growth with $\bar{\sigma}_z$ the homoskedastic component. Following, e.g., Fernández-Villaverde et al. (2011b) and Caldara et al. (2012), we specify the heteroskedastic component, $\sigma_{z,t}$, as

$$\sigma_{z,t} = \rho_\sigma \sigma_{z,t-1} + \tau \varepsilon_{\sigma_{z,t}}, \quad \varepsilon_{\sigma_{z,t}} \sim \mathcal{N}(0, 1) \quad (4.15)$$

¹⁰See also Swanson (2013).

¹¹As noted by Bansal and Yaron (2004, p. 1502), in an endowment economy with recursive preferences and stochastic volatility, better long-run growth prospects leads to a rise in the wealth-consumption and the price-dividend ratios. Rudebusch and Swanson (2012, p. 108) incorporate both real and nominal long-run risk in a production economy with recursive preference, and find long-run nominal risk improves the model's ability to fit the data.

where $|\rho_\sigma| < 1$ and τ is the standard deviation of $\varepsilon_{\sigma,t}$. The model is closed by the market clearing condition

$$Y_t = C_t + I_t \quad (4.16)$$

setting consumption and investment equal to output in each period.

The solution is characterized by the intratemporal labor supply/productivity condition equalizing the utility cost of marginally increasing labor supply to the utility value of the additional consumption obtained therewith

$$\frac{\psi}{1 - N_t} = \frac{1}{C_t} (1 - \alpha) \frac{Y_t}{N_t} \quad (4.17)$$

and the intertemporal Euler equation

$$E_t [m_{t+1} (1 + r_{t+1})] = 1 \quad (4.18)$$

where the real risky rate r_t comes from combining firms' profit and households' utility maximization

$$1 + r_t = \alpha K_{t-1}^{\alpha-1} (e^{z_t} N_t)^{1-\alpha} + 1 - \delta = r_t^K + 1 - \delta \quad (4.19)$$

and where m_{t+1} the stochastic discount factor or pricing kernel is given by

$$m_{t+1} \equiv \frac{\partial V_t / \partial C_{t+1}}{\partial V_t / \partial C_t} = \beta \frac{C_t}{C_{t+1}} \frac{\exp(\frac{\gamma}{2} V_{t+1})}{E_t [\exp(\frac{\gamma}{2} V_{t+1})]} \quad (4.20)$$

with V_t implicitly evaluated at the maximum.

As the economy is nonstationary, growing at the rate a_t , we detrend output, consumption, investment, capital stock and value function to stationarize the model. This is achieved by dividing all nonstationary variables but the value function, which must be detrended differently, by the contemporaneous level of productivity e^{Z_t} .¹² Labor supply N_t and leisure $1 - N_t$ as well as the returns r_t and r_t^K are stationary and therefore do not need to be transformed. Stationary variables will be denoted by lower case letters.

Reexpressing the pricing kernel in terms of stationary variables, the stochastic trend or long-run risk can be seen directly in the pricing kernel

$$m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\bar{a} + \bar{\sigma}_z e^{\sigma_{z,t+1}} \varepsilon_{z,t+1})} \frac{\exp\left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \bar{\sigma}_z e^{\sigma_{z,t+1}} \varepsilon_{z,t+1})\right]\right)}{E_t \left[\exp\left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \bar{\sigma}_z e^{\sigma_{z,t+1}} \varepsilon_{z,t+1})\right]\right) \right]} \quad (4.21)$$

where the stochastic trend, $\bar{\sigma}_z e^{\sigma_{z,t+1}} \varepsilon_{z,t+1}$, enters the kernel through the twisted continuation utility as well as through the stochastic discount factor that would obtain under expected utility, $\beta \frac{C_t}{C_{t+1}}$, and thereby explicitly relates the volatility process $\sigma_{z,t+1}$ to the pricing kernel as in, e.g., Bansal et al. (ming).

To analyze asset prices, we append the model with the following additional as-

¹²See the appendix for details.

set pricing variables: the real risk-free rate $1 + r_t^f \equiv E_t(m_{t+1})^{-1}$, the squared conditional market price of risk—the ratio of the conditional variance of the pricing kernel to square of its conditional mean¹³— $cmpr_t \equiv \frac{(E_t[(m_{t+1} - E_t m_{t+1})^2])^{\frac{1}{2}}}{E_t m_{t+1}}$ that measures the excess return the household demands for bearing an additional unit of risk, the expected (ex ante) risk premium $erp_t \equiv E_t(r_{t+1} - r_t^f)$, and the (ex post) risk premium $rp_t = r_t - r_{t-1}^f$ as the difference between the risky and risk-free rate.

4.4 Perturbation Solution and Risk Adjustment Channel

We solve the model in the foregoing section with a third order perturbation. As shown by Caldara et al. (2012), low order local approximations via perturbation methods can solve models such as ours quickly with a degree of accuracy comparable to global methods. Moreover, as at least a third order approximation is necessary for the analysis of time-varying shifts in risk premia and related measures at the heart of our analysis, we solve the model to third order. We use the nonlinear moving average perturbation derived in Lan and Meyer-Gohde (2013d) as it delivers stable impulse responses and simulations and, as we shall show, enables analytical calculation and risk decomposition of moments.

For the implementation of the nonlinear moving average perturbation, we collect the (stationarized) equilibrium conditions into a vector of functions

$$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)] \quad (4.22)$$

where y_t is the vector of the endogenous variables, and ε_t the vector of the exogenous shocks, assuming the function f in (4.22) is sufficiently smooth and all the moments of ε_t exist and finite¹⁴.

The solution to (4.22) is a time-invariant function y , taking as its state variable basis the infinite sequence of realized shocks, past and present, and indexed by the perturbation parameter $\sigma \in [0, 1]$ scaling the distribution of future shocks

$$y_t = y(\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots) \quad (4.23)$$

¹³We square the market price of risk to bestow it with the differentiability at the deterministic steady state necessary for our perturbation approach

¹⁴See for example, Judd (1998, ch. 13) and Jin and Judd (2002) for a complete characterization of these assumptions. While the normal distribution for shocks we choose is at odds with Jin and Judd's (2002) assumption of bounded support, Kim et al. (2008) dispute the essentiality of this assumption, lending support to our distribution choice

Table 4.1 Parameter Values: Common to All Three Calibrations

Parameter	β	ψ	χ	α	δ	\bar{a}	ρ_σ	τ
Value	0.9926	2.9869	100	0.339	0.021	0.004	0.9	0.15

See Tallarini (2000) and the main text.

Assuming normality of all the shocks and setting $\sigma = 1$ as we are interested in the stochastic model, the third order approximation—a Volterra expansion, see Lan and Meyer-Gohde (2013d)—of (5.31), takes the form

$$y_t^{(3)} = \bar{y} + \frac{1}{2} y_{\sigma^2} + \frac{1}{2} \sum_{i=0}^{\infty} (y_i + y_{\sigma^2,i}) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (4.24)$$

$$+ \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k,j,i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i})$$

where \bar{y} denotes the deterministic steady state of the model, at which all the partial derivatives y_{σ^2} , $y_{\sigma^2,i}$, y_i , $y_{j,i}$ and $y_{k,j,i}$ are evaluated. (5.32) is naturally decomposed into order of nonlinearity and risk adjustment— y_i , $y_{j,i}$ and $y_{k,j,i}$ capture the amplification effects of the realized shocks $(\varepsilon_t, \varepsilon_{t-1}, \dots)$ in the policy function (5.31) at first, second and third order respectively. The two partial derivatives with respect to σ , y_{σ^2} and $y_{\sigma^2,i}$ adjust the approximation for future risk.¹⁵ While y_{σ^2} is a constant adjustment for risk and a linear function of the variance of future shocks¹⁶, $y_{\sigma^2,i}$ varies over time, interacting the linear response to realized shocks with the variance of future shocks essentially adjusting the model for time variation in the conditional volatility of future risk.

4.5 Calibration

We select three calibrations for the numerical analysis of the model. For the baseline calibration, most of the parameter values are taken from Tallarini (2000) and are listed below.

The discount factor $\beta = 0.9926$ generates an annual interest rate of about 3 percent. The capital share $\alpha = 0.339$ matches the ratio of labor share to national income. The depreciation rate $\delta = 0.021$ matches the ratio of investment to output. The parameter χ is set to 100, translating to a relative risk aversion parameter with respect to con-

¹⁵More generally, a constant term, y_{σ^3} , at third order adjusts (5.32) for the skewness of the shocks. See Andreasen (2012b). As we assume all the shocks are normally distributed, y_{σ^3} is zero and not included in (5.32) and the rest of our analysis.

¹⁶See, Lan and Meyer-Gohde (2013d, p. 13) for the derivation of this term.

Table 4.2 Parameter Values: Calibrating Homoskedastic Volatility

	Baseline	Constant Volatility	Expected Utility	Extended Model
$\overline{\sigma}_a$	0.009824769	0.011588754	0.0115	0.0225

$\overline{\sigma}_a$ calibrated to keep the standard deviation of $\Delta \ln(c) = 0.0055$

sumption of about 25 following Tallarini (2000).¹⁷ The labor supply parameter ψ is chosen such that labor in the deterministic steady state, \overline{N} , is 0.2305 to align with the mean level of hours in data following Tallarini (2000). With β , χ , and ψ as above, $\gamma = -0.3676$ in line with Tallarini (2000).

For the parameters of the volatility shock, the literature varies in the range of the persistence— ρ_σ , from 0.9, Caldara et al. (2012) and Bidder and Smith (2012), to 0.95, Fernández-Villaverde and Rubio-Ramírez (2010a), and to 0.99 or 1, Andreasen (2012b) and Justiniano and Primiceri (2008)—and in the range of its instantaneous standard deviation— τ , from 0.01, Andreasen (2012b) and Justiniano and Primiceri (2008), to 0.1, Fernández-Villaverde and Rubio-Ramírez (2010a), and to 0.15, Bidder and Smith (2012). We follow the parameterization of Bidder and Smith (2012), implying a cumulative variance comparable to the value in Fernández-Villaverde and Rubio-Ramírez (2010a, p. 20), that “generates changes in volatility similar to the ones observed in the [post-war] U.S.” Following Tallarini (2000), we adjust the homoskedastic component of the standard deviation of productivity growth to match the standard deviation of (log) consumption growth.

While still allowing preferences to be recursive, the constant volatility calibration shuts down stochastic volatility by setting $\rho_\sigma = \tau = 0$, this enables direct comparison with Tallarini’s (2000) results. In addition, by comparing with the results from the baseline calibration, this exercise helps identify the contribution of the stochastic volatility, by itself and/or in interaction with recursive preferences, to the model. The expected utility calibration shuts stochastic volatility down and is implemented by setting $\chi = 1$ (equivalently, $\gamma = 0$).

We will use all the three calibrations to analyze the contributions of recursive preferences and stochastic volatility to the model’s performance in matching empirical macroeconomic and asset pricing statistics.

¹⁷Tallarini (2000) gives $\frac{\psi+\chi}{1+\psi}$ for this measure of risk aversion. Swanson (2013) incorporating the active labor margin as in Swanson (2012), gives $\frac{\chi}{1+\psi}$ for this same measure of risk aversion. At our calibration, these measures correspond to 25.831 and 25.08 respectively.

4.6 Theoretical Moments

In this section, we derive the theoretical moments of the third order approximation (5.32). The nonlinear moving average policy function (5.31) and its third order approximation (5.32) both map exogenous shocks directly into endogenous variables. The moments of endogenous variables can therefore be computed directly as they are functions of the known moments of exogenous shocks. We further decompose the theoretical variance, disentangling the individual contributions of the risk adjustment and amplification channels to the total variance. Note that throughout the derivation of theoretical moments, we assume normality of the exogenous shocks and all the approximated variables are covariance stationary.¹⁸

By contrast, the state space perturbation policy function and its nonlinear approximations map the endogenous variables into themselves. Computing the m -th theoretical moment of such a nonlinear approximations of n -th order, for example, requires the knowledge of higher (than m -th) moments of endogenous variables that are in general nonlinear functions of the approximations up to and including n -th order. To this end, the calculation results in an infinite regression in the moments of endogenous variables. While theoretical moments of nonlinear state space perturbation approximations are in general not available, there are attempts in recent literature. Andreasen et al. (2013) calculate theoretical moments by pruning the nonlinear approximations, such that the higher (than m -th) moments are functions of approximations lower than the current order of approximation, and therefore computable given the results from all lower orders.¹⁹

4.6.1 Mean

The mean (first moment) of the third order approximation (5.32) is straightforward to calculate. Applying the expectations operator to (5.32) yields

$$E \left[y_t^{(3)} \right] = \bar{y} + \frac{1}{2} y \sigma^2 + \frac{1}{2} \sum_{j=0}^{\infty} y_{j,j} E [\varepsilon_t \otimes \varepsilon_t] \quad (4.25)$$

The last term in (5.32) vanishes as the triple Kronecker product in expectation is the columnwise vectorization of the third moment of the exogenous shocks, equal to zero

¹⁸While removing normality does not disable the calculation of theoretical moments, the derivation will be more complicated as additional terms involving skewness and higher (up to fifth) moments of the shocks emerge. See Lan and Meyer-Gohde (2013d) for proof of the covariance stationarity.

¹⁹As shown in Lan and Meyer-Gohde (2013b), nonlinear moving average perturbations at the third order differ from their pruned state space counterparts in that they are centered around and correct the first derivative of the policy function for the stochastic steady state implied by the order of approximation, leading to accuracy gains in a mean squared sense.

under normality. Likewise, the Kronecker product in expectation is the columnwise vectorization of the second moment of the exogenous shocks. Only the contemporaneous variance appears because the shock vector is assumed serially uncorrelated. The other two terms containing ε_{t-i} in (5.32) also disappear as the shock is mean zero. From a different perspective, the deterministic steady state is the mean of the zeroth order approximation where all shocks, past, present and future are zero. It remains the mean in a first order approximation, as the exogenous shocks are mean zero (first moment is zero). At second order, the second moments of the shocks are included—both past and present (in the term $\sum_{j=0}^{\infty} y_{j,j} E[\varepsilon_t \otimes \varepsilon_t]$) as well as future shocks (in the term y_{σ^2})—which are assumed nonzero, generating an adjustment from the deterministic steady state. When the approximation moves to the third order, the calculation of the mean of (5.32) would be accordingly adjusted for the first three moments of all the realized and future shocks, but the mean zero and normality assumptions render the first and third moments of the shocks zero, thus leaving the first moment at third order identical to its value from a second order approximation.

4.6.2 Variance and Autocovariances

While we could conceivably compute the second moments (variance and autocovariances) of (5.32) using the Volterra expansion directly, it would be a rather complicated operation on the products of multi-layered infinite summation of coefficients. As an alternative, we use the recursive expression of (5.32) derived in Lan and Meyer-Gohde (2013b) to compute the second moments.

Computing the second moments using the recursive expression of (5.32), we need to proceed sequentially through the orders of approximation and exploit the linearly recursive (in order) structure of the solution. That is, the second moments of the approximation at any order can always be expressed as the sum of the second moments of the approximation of the previous order and the second moments of all the previous order increments (the difference between two approximations of adjacent order, subtracting the constant risk adjustment of the higher order). In other words, the embedded decomposition into order of approximation in the nonlinear approximations of the policy function (5.31) is preserved in its second moments.

The first order approximation of (5.31) takes the form of a linear moving average,

$y_t^{(1)} = \bar{y} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i}$, and can be expressed recursively as²⁰

$$y_t^{(1)} - \bar{y} = \alpha \left(y_t^{(1)state} - \bar{y}^{state} \right) + \beta_0 \varepsilon_t \quad (4.26)$$

where the difference $y_t^{(1)} - \bar{y}$ is the deviation of the first order approximation with respect to the deterministic steady state, and identical to the first order increment

$$dy_t^{(1)} \equiv y_t^{(1)} - \bar{y} \quad (4.27)$$

which captures the addition to the approximation contributed by the time varying terms of the current, here first, order of approximation, as \bar{y} is the zeroth order approximation²¹ and the constant risk adjustment of first order, y_{σ} , is zero. In addition

$$E \left[dy_{t-1}^{(1)} \varepsilon_t' \right] = 0 \quad (4.28)$$

as the current shock is not correlated with the endogenous variables in the past. Under the orthogonality condition (4.28), the sequence of autocovariances of endogenous variables or, at this order equivalently, of the first order increment $\Gamma_j^{y^{(1)}} = \Gamma_j^{(1)} = E \left[dy_t^{(1)} dy_{t-j}^{(1)'} \right]$, solves the following Lyapunov equation

$$\Gamma_j^{y^{(1)}} = \alpha \Gamma_j^{y^{(1)}} \alpha' + \beta_0 E[\varepsilon_t \varepsilon_{t-j}] \beta_0' \quad (4.29)$$

The second order approximation of the policy function (5.31) captures the amplification effects of the realized shocks up to second order, and the constant risk adjustment for future shocks

$$y_t^{(2)} = \bar{y} + \frac{1}{2} y_{\sigma^2} + \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (4.30)$$

Defining the second order increment

$$dy_t^{(2)} \equiv y_t^{(2)} - y_t^{(1)} - \frac{1}{2} y_{\sigma^2} \quad (4.31)$$

which more clearly illustrates the notion of increment we use here: the addition the approximation contributed by time varying components of current order (or the difference between the current and previous order of approximation, here $y_t^{(2)} - y_t^{(1)}$, less the additional constant contributed by the current order, here $\frac{1}{2} y_{\sigma^2}$). With this notion, the second order approximation (4.30) can be considered as the sum of first order approximation, the constant risk correction term and second order increment

$$y_t^{(2)} = y_t^{(1)} + \frac{1}{2} y_{\sigma^2} + dy_t^{(2)} \quad (4.32)$$

The above decomposition of second order approximation naturally passes on to its

²⁰See Lan and Meyer-Gohde (2013b). This is, of course, an standard result for linear models. Compare, e.g., the state space representations of Uhlig (1999) with the infinite moving average representations of Taylor (1986).

²¹This is the terminology in Anderson et al. (2006, p. 17) and Borovicka and Hansen (2012, p. 22).

moments — Starting with the mean, taking expectation of (4.32) yields

$$Ey_t^{(2)} = Ey_t^{(1)} + \frac{1}{2}y_{\sigma^2} + E dy_t^{(2)} \quad (4.33)$$

Therefore the mean of second order approximation is a sum of the mean of first order approximation, i.e., the deterministic steady state, the constant risk correction term, and the mean of the second order increment. Likewise, the second moments of the second order approximation can be expressed as the sum of the second moments of the first order approximation and those of the order increment. We summarize the results for a second order approximation in the following proposition

Proposition 4.6.1. *Assuming the exogenous shocks are normally distributed, the j 'th autocovariance of the second order approximation (4.30) is of the form*

$$\Gamma_j^{y^{(2)}} = \Gamma_j^{y^{(1)}} + \Gamma_j^{(2)} \quad (4.34)$$

where

$$\Gamma_j^{y^{(2)}} = E \left[\left(y_t^{(2)} - Ey_t^{(2)} \right) \left(y_{t-j}^{(2)} - Ey_{t-j}^{(2)} \right)' \right] \quad (4.35)$$

$$\Gamma_j^{y^{(1)}} = \Gamma_j^{(1)} = E \left(dy_t^{(1)} dy_{t-j}^{(1)'} \right) \quad (4.36)$$

$$\Gamma_j^{(2)} = E \left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_{t-j}^{(2)} \right)' \right] \quad (4.37)$$

Proof. See the appendices. \square

The second order increment $dy_t^{(2)}$ can also be expressed recursively.²² With that recursive expression in hand, the unknown $E dy_t^{(2)}$ in (4.33) and $\Gamma_j^{(2)}$ in (4.34) can be obtained by solving some standard linear matrix equations and an appropriate Lyapunov equation. The details are relegated to the appendices.

Similarly, to compute the second moments of endogenous variables using the third order approximation (5.32), we define the third order increment

$$dy_t^{(3)} \equiv y_t^{(3)} - y_t^{(2)} \quad (4.38)$$

which is merely the difference between the third and second order approximations, as the third order approximation adds no additional constant terms under normality. We summarize the resulting second moment calculations at third order in the following proposition

Proposition 4.6.2. *Assuming the exogenous shocks are normally distributed, the j 'th autocovariance of the third order approximation (5.32) takes the form*

$$\Gamma_j^{y^{(3)}} = \Gamma_j^{y^{(2)}} + \Gamma_j^{(3)} + \Gamma_j^{(1),(3)} + \left(\Gamma_j^{(1),(3)} \right)' \quad (4.39)$$

²²See, again, Lan and Meyer-Gohde (2013b).

where

$$\Gamma_j^{y^{(3)}} = E \left[\left(y_t^{(3)} - E y_t^{(3)} \right) \left(y_{t-j}^{(3)} - E y_{t-j}^{(3)} \right)' \right] \quad (4.40)$$

$$\Gamma_j^{(3)} = E \left(dy_t^{(3)} dy_{t-j}^{(3)'} \right) \quad (4.41)$$

$$\Gamma_j^{(1),(3)} = E \left(dy_t^{(1)} dy_{t-j}^{(3)'} \right) \quad (4.42)$$

and $\Gamma_j^{y^{(2)}}$ is as defined in Proposition 4.6.1.

Proof. See the appendices. \square

$\Gamma_j^{y^{(3)}}$ is the j 'th autocovariance of endogenous variables computed using the third order approximation (5.32), $\Gamma_j^{(3)}$ the j 'th autocovariance of the third order increment $dy_t^{(3)}$, and $\Gamma_j^{(1),(3)}$ the j 'th autocovariance between the first and the third order increments $dy_t^{(1)}$ and $dy_t^{(3)}$. Analogous to (4.34) in Proposition 4.6.1, (4.39) decomposes the second moments into order of approximation: When the approximation moves to the third order, the second moments of endogenous variables are those computed using second order approximation (4.30), adjusted by the second moments of $dy_t^{(3)}$ itself and the interaction with the first order increment $dy_t^{(1)}$.

With the recursive form of the third order increment $dy_t^{(3)}$,²³ the two unknown quantities, $\Gamma_j^{(3)}$ and $\Gamma_j^{(1),(3)}$, in (4.39) for calculating the covariance matrices of the third order approximation can be computed by formulating appropriate Lyapunov equations. The details are in the appendices.

4.6.3 A Variance Decomposition

The third order approximation, (5.32), decomposes naturally into orders of nonlinearity and risk adjustment. This dissects the individual contributions of the sequence of realized shocks and future shocks and a variance decomposition can be accordingly derived to analyze the composition of the volatility of endogenous variables.

Let $y_t^{(3)risk} \equiv \frac{1}{2} y_{\sigma^2} + \frac{1}{2} \sum_{i=0}^{\infty} y_{\sigma^2,i} \varepsilon_{t-i}$ denote the risk adjustment channel, with a constant risk adjustment at second order ($\frac{1}{2} y_{\sigma^2}$) and a time-varying risk adjustment channel at third order ($\frac{1}{2} \sum_{i=0}^{\infty} y_{\sigma^2,i} \varepsilon_{t-i}$) and $y_t^{(3)amp}$ collect all the other terms in the third order approximation (5.32) capturing the amplification effects, we can rewrite (5.32) as

$$y_t^{(3)} \equiv y_t^{(3)risk} + y_t^{(3)amp} \quad (4.43)$$

²³See, again, Lan and Meyer-Gohde (2013b).

Centering the previous equation around its mean,²⁴ multiplying the resulting expression with its transposition and applying the expectations operator yields the following variance decomposition

Proposition 4.6.3. *Assuming the exogenous shocks are normally distributed, the covariance of the third order approximation (5.32) takes the form*

$$\Gamma_0^{y^{(3)}} = \Gamma_0^{y^{(3)risk}} + \Gamma_0^{y^{(3)risk,amp}} + \Gamma_0^{y^{(3)amp}} \quad (4.44)$$

where

$$\Gamma_0^{y^{(3)risk}} = E \left[\left(y_t^{(3)risk} - E y_t^{(3)risk} \right) \left(y_t^{(3)risk} - E y_t^{(3)risk} \right)' \right] \quad (4.45)$$

$$\Gamma_0^{y^{(3)amp}} = E \left[\left(y_t^{(3)amp} - E y_t^{(3)amp} \right) \left(y_t^{(3)amp} - E y_t^{(3)amp} \right)' \right] \quad (4.46)$$

$$\Gamma_0^{y^{(3)risk,amp}} = E \left[\left(y_t^{(3)amp} - E y_t^{(3)amp} \right) y_t^{(3)risk'} \right] + \left(E \left[\left(y_t^{(3)amp} - E y_t^{(3)amp} \right) y_t^{(3)risk'} \right] \right)' \quad (4.47)$$

Proof. See the appendices. \square

The variance of the endogenous variables, $\Gamma_0^{y^{(3)}}$, can thus be expressed as the sum of $\Gamma_0^{y^{(3)risk}}$ that stores the variations come from the time-varying risk adjustment channel alone, $\Gamma_0^{y^{(3)amp}}$ that stores the variations come from the amplification channels of all three orders and $\Gamma_0^{y^{(3)risk,amp}}$ that stores the variations come from the interaction between the two types of channels.

Both $y_t^{(3)risk}$ and $y_t^{(3)amp}$ can be expressed recursively. With those recursive expressions, $\Gamma_0^{y^{(3)risk}}$ and $\Gamma_0^{y^{(3)amp}}$ can be computed by formulating appropriate Lyapunov equations (See the appendices for details). As $\Gamma_0^{y^{(3)}}$ is already known from Proposition 4.6.2, $\Gamma_0^{y^{(3)risk,amp}}$ can be computed by subtracting $\Gamma_0^{y^{(3)risk}}$ and $\Gamma_0^{y^{(3)amp}}$ from $\Gamma_0^{y^{(3)}}$.

4.6.4 Simulated Moments

As an alternative to the theoretical moments, we can simulate the third order approximation (5.32) and compute the moments of the simulated series to analyze the statistical implications of the model. Lan and Meyer-Gohde (2013d) show that nonlinear approximation of the policy function (5.31) preserve the stability of the linear approximation or first order approximation and, hence, does not generate explosive time paths in simulations.

²⁴Note $E y_t^{(3)risk} = \frac{1}{2} y_{\sigma^2}$ and $E y_t^{(3)amp} = \bar{y} + \frac{1}{2} \sum_{j=0}^{\infty} y_{j,j} E [\varepsilon_t \otimes \varepsilon_t]$.

Simulation methods for moment calculations are, however, not always feasible for state space perturbations. Aruoba et al. (2006), Fernández-Villaverde and Rubio-Ramírez (2006) and Kim et al. (2008) note that higher order Taylor approximations to state space perturbation policy function can be potentially explosive in simulations. Truncation of the distribution from which exogenous shocks are drawn or the application of pruning schemes, like proposed by Kim et al. (2008) for a second order approximation,²⁵ can prevent such behavior. While this imposes stability on simulations of higher order approximations, pruning is an ad hoc procedure as noted by Lombardo (2010) and potentially distortive even when the simulation is not on an explosive path (See, Den Haan and De Wind (2012a)). Though this might give rise to reasonable doubts regarding the accuracy and validity of moments calculated using perturbations, we will show that this is not the case with our nonlinear moving average.

As (5.32) generates stable time paths, moments computed by simulating (5.32) should asymptotically converge to their theoretical counterparts.

Figure 4.1 is an example of this check. It depicts the evolution path of the density of the simulated variance of the pricing kernel in the model described in Section 5.2 under the baseline calibration. Densities of the simulated variance of the pricing kernel are calculated using a kernel density estimation and 100 simulations at the indicated length. The theoretical variance, denoted by the red dashed line, is 0.0666 and all densities are in general centered around this value. The distributions of simulated variance are more dispersed in short-run simulations, tightening up to the theoretical value as the length increases consistent with asymptotic convergence of the simulated moments to their theoretical counterparts we calculated above.

4.7 Analysis of the Baseline Model

In this section, we report the performance under different calibrations of the model approximated to third order. We present impulse responses using the method of Lan and Meyer-Gohde (2013d) to shocks in productivity growth and its volatility for both macroeconomic and asset pricing variables. We then proceed to the moments and the results of the variance decomposition introduced in section 4.6.3 to identify and quantify the individual contribution from the time-varying risk adjustment channel to the total variation. Finally, we cast doubt on the efficacy of stochastic volatility in aiding the model ability in attaining the Hansen-Jagannathan bounds.

²⁵See Lan and Meyer-Gohde (2013b) for an overview and comparison of pruning algorithms at second and third order and their relation to our nonlinear moving average.

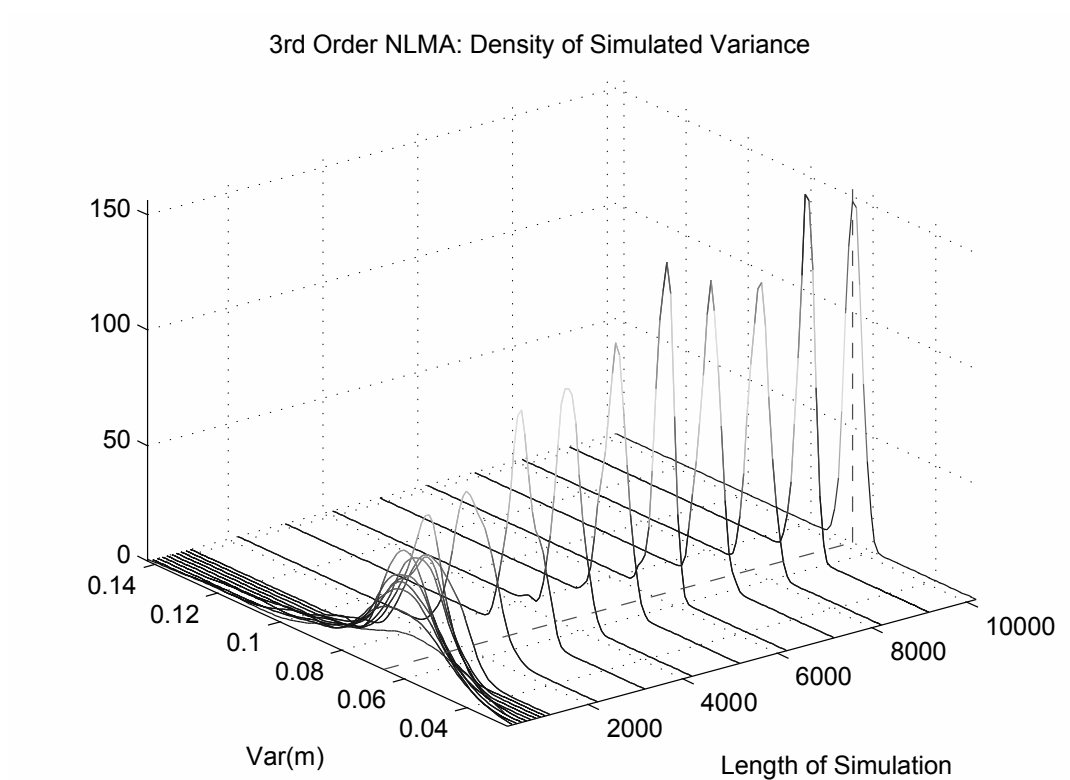


Fig. 4.1 Monte Carlo Consistency of Moment Calculations, Example of m_t , Baseline Model of Section 5.2

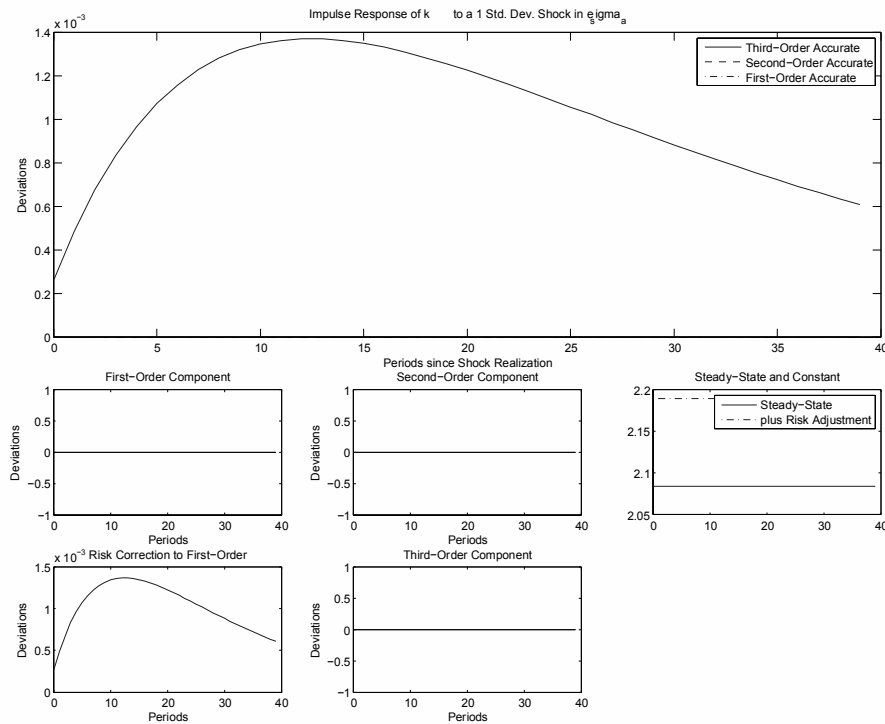


Fig. 4.2 Capital IRF: Volatility Shock, Baseline Model of Section 5.2

4.7.1 Impulse Responses and Simulations

We analyze the impulse responses to shocks in productivity growth and shock in its volatility for macroeconomic and asset pricing variables. We also simulate the conditional market price of risk under stochastic volatility and with growth shocks of constant variance to observe the change in the variations of this variable under conditional heteroskedasticity.

Figure 5.1 depicts the impulse response and its contributing components for capital to a positive, one standard deviation shock in volatility, i.e., in $\varepsilon_{\sigma_{z,t}}$. The upper panel displays the impulse responses at first, second and third order as deviations from their respective (non)stochastic steady states (themselves in the middle right panel). In the middle left panel and the middle column of panels in the lower half of the figure, the contributions to the total impulse responses from the first, second and third order amplification channels, that is, y_i , $y_{i,i}$ and $y_{i,i,i}$ in the third order approximation (5.32), are displayed. Notice that there is no response in these amplification channels. All

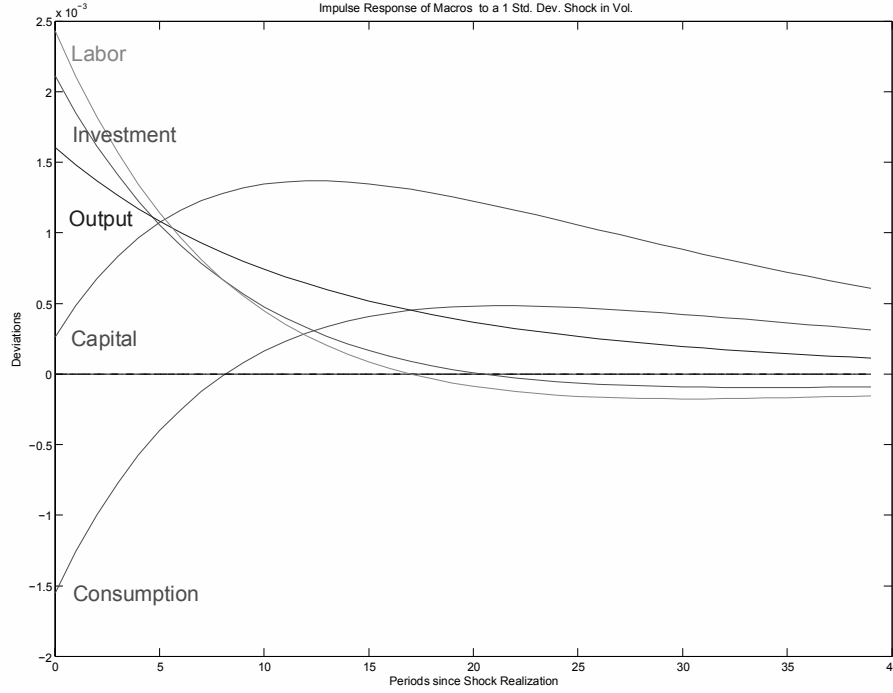


Fig. 4.3 Macro IRFs: Volatility Shock, Baseline Model of Section 5.2

responses to this volatility shock come from the lower left panel of the figure where the time-varying risk adjustment channel $y_{\sigma^2,i}$ is displayed. In other words, for capital, a volatility shock by itself propagates solely through the time-varying risk adjustment channel.

Capital responds positively to a positive volatility shock. This captures the household's precautionary reaction to the widening of the distribution of future shocks.²⁶ Our risk-averse household accumulates a buffer stock in capital to insure itself against the increased future risk of productivity growth shocks from a more dispersed distribution.

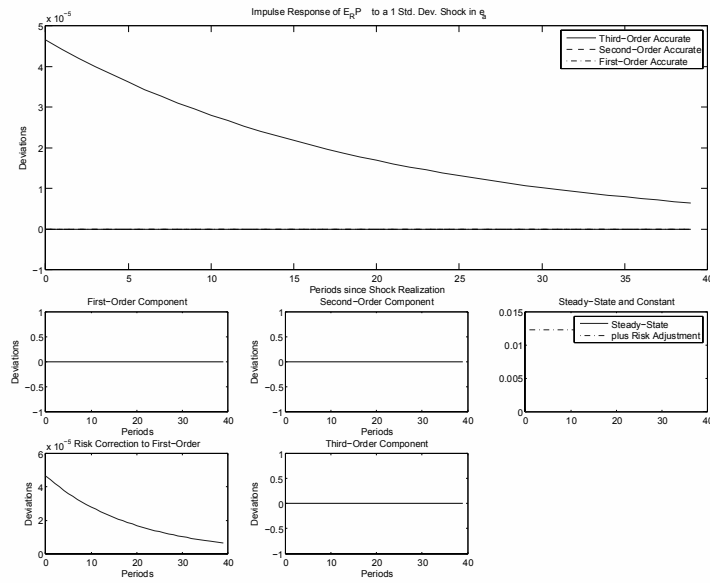
Figure 5.3 displays the responses of macroeconomic variables as deviations from their risk corrected steady states to a positive, one standard deviation volatility shock. The household accumulates a buffer stock of capital by increasing current investment on impact of the shock. As the allocation has not changed, the household finances this

²⁶See also Fernández-Villaverde and Rubio-Ramírez (2010a) and van Binsbergen et al. (2012) for precautionary savings behavior in DSGE perturbation.

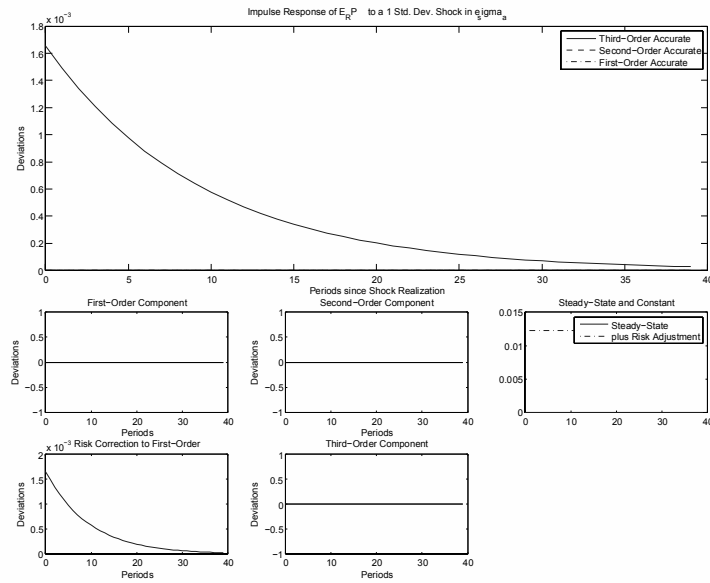
investment through a decrease in current consumption, resulting in an increase in the marginal utility of consumption. The intratemporal labor supply equation (4.17) implies this increased marginal utility of consumption leads to an increase in the marginal utility of leisure, and therefore a decrease in time spend on leisure. The increased labor effort, with the capital stock being fixed on impact as it is a state variable and with the productivity having not changed,²⁷ translates into an increase in current output partially offsetting the costs borne by consumption of the increased investment for the buffer stock of capital. Thus, this model predicts a boom in economic activity following an increase in risk, as firms produce and households work to accumulate the necessary buffer stock. A richer model of frictional investment that, for example, includes variable capacity utilization, capital adjustment cost and consumption habit formation can overturn this result, as discussed in section 4.8. While the impulse responses for the macroeconomic variables are not pictured with their contributing components, responses of these variables to a volatility shock come solely from the time-varying risk adjustment channel. The volatility shock is persistent but not permanent. As the shock dies out and productivity shocks fail to materialize from their widened distribution, the household winds down its buffer stock of capital by increasing consumption and leisure, leading to a fall in output and investment.

Figure 4.4 depicts the impulse responses and their contributing components for the expected risk premium to positive, one standard deviation shocks in productivity and its volatility, i.e., in $\varepsilon_{z,t}$ and $\varepsilon_{\sigma_{z,t}}$ (Figure 4.4a and 4.4b respectively). Firstly, note that both the volatility and productivity growth shock propagate solely through $y_{\sigma^2,i}$, the time-varying risk adjustment channel of this variable and there are no responses in the amplification channels of any of the three orders. Second, the response of the expected risk premium to the volatility shock is almost two orders magnitude larger than that to the productivity growth shock, implying the overall majority of variations in this variable is driven solely by volatility shocks with the contribution from the productivity growth shock to the total variation negligible. Moreover, the positive response of the expected risk premium to an increase in volatility highlights the role of long run risk in asset pricing as noted by Bansal and Yaron (2004), Bansal (2008) and Bansal et al.'s (ming) — risks and volatility in asset prices are driven by those in economic fundamentals. An increase of volatility in long run productivity growth therefore drives up risks in asset prices and makes holding asset riskier. Household thereby demands a higher compensation for doing so. I.e., an increase in volatility of

²⁷Note that, it is the distribution governing future productivity shocks that is being shocked here, not the level of productivity itself.



(a) Growth Shock



(b) Volatility Shock

Fig. 4.4 Expected Risk Premium IRFs: Volatility and Growth Shocks, Baseline Model of Section 5.2

Table 4.3 Mean Comparison

Variable	<i>Det.S.S.*</i>	Baseline Cal.	Constant Vol. Cal.	Tallarini (2000)
$\log(k)$	2.0841	2.1373	2.1581	2.1584
i	0.2002	0.2106	0.2146	0.2160
$\log(c)$	-0.5672	-0.5542	-0.5491	-0.5499
$\log(y)$	-0.2649	-0.2417	-0.2326	-0.2319
$\log(N)$	-1.4675	-1.4597	-1.4566	-1.4563
R^f	1.1493	1.0470	1.0070	1.011
R	1.1493	1.0532	1.0156	1.022

* The deterministic steady state value
See Table 5 and 8, Tallarini (2000).

long run risk carries a positive risk premium.

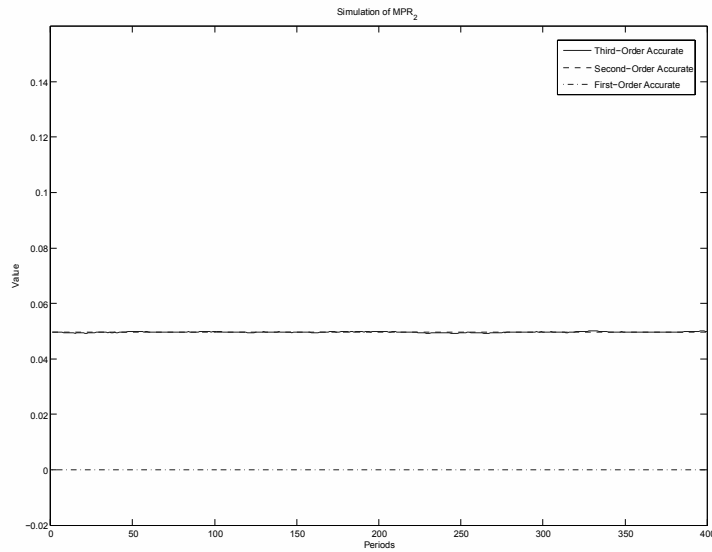
Figure 4.5 depicts the simulated time paths of the squared conditional market price of risk²⁸ under the constant volatility and the baseline calibration of the model (Figure 4.5a and 4.5b respectively). When there is no volatility shock, the conditional market price exhibits minimal fluctuations along the simulation path. Adding stochastic volatility, however, induces a substantial amount of variations in this variable. This is consistent with the interpretation that volatility shocks are a source of conditional heteroskedasticity. The displayed time variation in the conditional market price of risk is roughly consistent with the empirical variations in the (lower bound of) market price of risk as measured over different periods of time the past 130 odd years (See, Cogley and Sargent (2008, p. 466)).

4.7.2 Moments Comparison

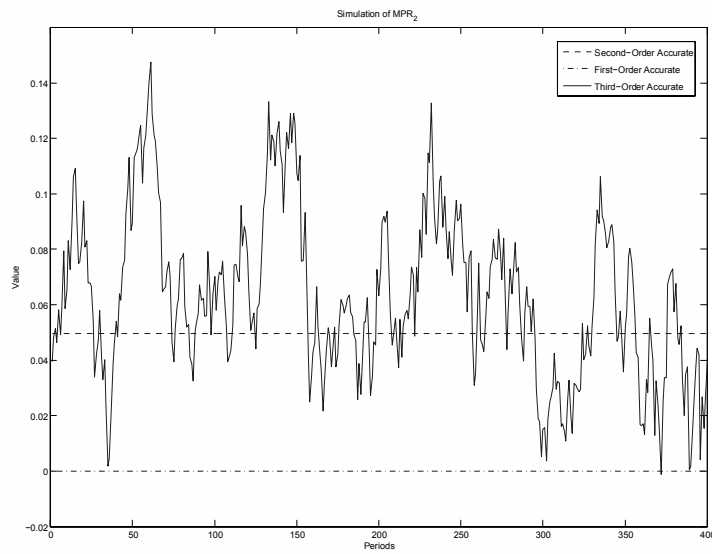
We compare the mean and standard deviations of the third order approximation (5.32) to those reported in Tallarini (2000) for his model and post-war U.S. data. The results of the variance decomposition in Section 4.6.3 are reported, allowing us to pin down the contribution from the time-varying risk adjustment channel to the total variance of the endogenous variables.

The third and fourth column of Table 4.3 report the theoretical means under the baseline and constant volatility calibration of the model. The last column displays the means of the constant volatility calibration as reported by Tallarini (2000) with his iterative modified linear quadratic approximation based on Hansen and Sargent

²⁸We square this variable to eliminate the kink at the deterministic steady state and make perturbation applicable.



(a) Constant Volatility Calibration



(b) Baseline (Stochastic Volatility) Calibration

Fig. 4.5 Simulated Squared Conditional Market Price of Risk, Baseline and Constant Volatility Model of Section 5.2

Table 4.4 Standard Deviation Comparison

Variable	Baseline Calibration	Constant Volatility Calibration	Tallarini (2000)	Data
$\Delta \log(c)$	0.0055	0.0055	0.0055	0.0055
$\Delta \log(y)$	0.0096	0.0100	0.0095	0.0104
$\Delta \log(i)$	0.0240	0.0223	0.0224	0.0279
$\log(c) - \log(y)$	0.0154	0.0150	0.0147	0.0377
$\log(i) - \log(y)$	0.0425	0.0404	0.0403	0.0649

See Table 7, Tallarini (2000).

(1995). By comparing the last two columns we observe, for both macroeconomic and asset pricing variables, our theoretical means are in line with those of Tallarini (2000).

In the presence of risks induced by the long run productivity growth shocks (with or without stochastic volatility), the means of macroeconomic quantities (reported in the first five rows of the third and fourth column) are uniformly larger than their corresponding deterministic steady state value (reported in the first five rows of the first column), reinforcing the interpretation of household's precautionary reaction to future shocks.²⁹ In contrast, the mean of risky and risk free rates (reported in the last two rows of the third and fourth column) are uniformly lower than their deterministic counterparts (reported in the last two rows of the first column). This follows directly from the increase in the mean of capital which reduces the average return on equity (risky rate) and consequentially the risk free rate as noted by Tallarini (2000).

The second and third column of Table 4.4 report the theoretical standard deviations of the third order approximation (5.32) under the baseline and constant volatility calibration of the model. Comparing to the standard deviations reported in the last two columns, our theoretical standard deviations are in line with those reported in Tallarini (2000), both model based and empirical.

Table 4.5 reports the results of the variance decomposition under the baseline (stochastic volatility) and the constant volatility calibration. For each calibration, the table reports the percentage contributions of the first order amplification channel $y_t^{(1)}$ and the time-varying risk adjustment channel $y_t^{(3)risk}$ to the total variance of the endogenous variables as the overall majority of variations come from these two channels. The second and third column report the decomposition results in absence of volatility shock and the last two columns in presence of volatility shock. For the conditional market price of risk and the expected risk premium, all variation comes from the time-varying

²⁹That the mean of higher order approximation of macroeconomic quantities captures precautionary reactions and hence are different from their deterministic steady state counterparts are also noted by Tallarini (2000), Juillard (2011) and Coeurdacier et al. (2011) in state space context.

Table 4.5 Variance Decomposition in Percentage

	Constant Volatility Calibration		Baseline Calibration	
	1st order amp.	risk adjustment	1st order amp.	risk adjustment
<i>cmpr</i>	0	100	0	100
<i>erp</i>	0	100	0	100
<i>rp</i>	109.03	0.60	80.68	8.51
$\log(k)$	96.35	0.02	75.08	1.37
<i>i</i>	95.28	0.02	57.19	30.25
$\log(c)$	96.65	0.01	75.88	2.62
$\log(y)$	95.03	0.04	44.54	36.99
$\log(N)$	97.82	0.01	66.25	18.61

For each calibration, the columns may not add up to 100 due to the omission of 2nd and 3rd order amplification and cross effects.

risk adjustment channel regardless of whether there is volatility shock. This is consistent with the impulse responses for the expected risk premium (Figure 4.4), where we observed that both the productivity growth and volatility shock propagate solely through the time-varying risk adjustment channel.

For the risk premium and macroeconomic variables, adding the volatility shock alters the composition of variance. In the absence of the volatility shock, the contribution of the time-varying risk adjustment channel is negligible and almost all variation comes from the first order amplification channel. Adding stochastic volatility, however, operationalizes the time-varying risk channel, as a large portion of variance now comes through changes in risk as measured by conditional heteroskedasticity. Since, for macroeconomic variables, actions in the time-varying risk adjustment channel can be explained by the risk-averse household's precautionary motives, this variance decomposition result implies that such motives account for a larger portion of variance in the presence of stochastic volatility than in the absence thereof. An presence of changing risk induces the pattern of precautionary behavior here—with investment, output, and labor driven substantially by risk adjustments—as the capital margin cannot be freely adjusted contemporaneously in response to shifts in the distribution of technology shocks, pushing the adjustment onto production, the other factor of production, labor, and the component of expenditure, investment, over which the household does not have a direct smoothing motive.

From a methodological point of view, in the absence of volatility shock, a first order linear approximation would thus appear sufficient for computing the theoretical variance of macroeconomic variables. However, theoretical variances need to be com-

puted using a third order approximation in the presence of stochastic volatility and for conditional asset pricing measures, as otherwise a large portion or all of the variance will be missed through the neglect of time varying risk adjustment and higher order amplification effects.³⁰

4.7.3 Stochastic Volatility and Hansen-Jagannathan Bounds

We evaluate the model's ability in attaining the Hansen-Jagannathan bounds under the three different calibrations. The bounds present an important empirical measure that depend on the first two moments of the pricing kernel for a model's ability of replicating asset pricing regularities. Contrary to studies in endowments models where the variance of the log consumption growth process is fixed exogenously, the variance of log consumption growth here is endogenous, deriving eventually from the productivity process. While adding stochastic volatility does *ceteris paribus* move the model closer to the Hansen-Jagannathan bounds, it does so at the cost of increasing the variance of the consumption process. Adjusting the homoskedastic component of productivity to hold the variance of log consumption growth constant, the move towards the Hansen-Jagannathan bounds is negated.

Figure 4.6 depicts the unconditional mean standard deviation pairs of the pricing kernel generated by the model under the three different calibrations. Under the baseline (stochastic volatility) and the constant volatility calibrations, the preferences are in recursive form. While the volatility of the kernel increases with the coefficient of relative risk aversion for atemporal wealth gambles (here from χ equals one to five, ten, twenty, thirty, forty, fifty, and one hundred), the unconditional mean of the kernel is left (essentially) unchanged as the elasticity of intertemporal substitution (EIS) is parameterized independent of risk aversion in recursive preferences, and the model approaches the Hansen-Jagannathan Bounds from below. The expected utility calibration generates a volatile pricing kernel at the cost of reducing its unconditional mean, as the EIS and risk aversion are inversely correlated in the expected utility, generating Weil's (1989) risk free rate puzzle. Figure 4.6a shows that given the same value of risk aversion, the calibration with stochastic volatility (baseline calibration) generates a more volatile pricing kernel than the constant volatility calibration. In other words, to generate certain amount of volatility in the pricing kernel, the model with volatility shock appears to need less risk aversion than the model without. This is achieved, how-

³⁰This provides insight, and a proviso in the presence of stochastic volatility, into the practice of computing macro variables using first and conditional asset pricing measures with third order approximations as in Rudebusch and Swanson (2012).

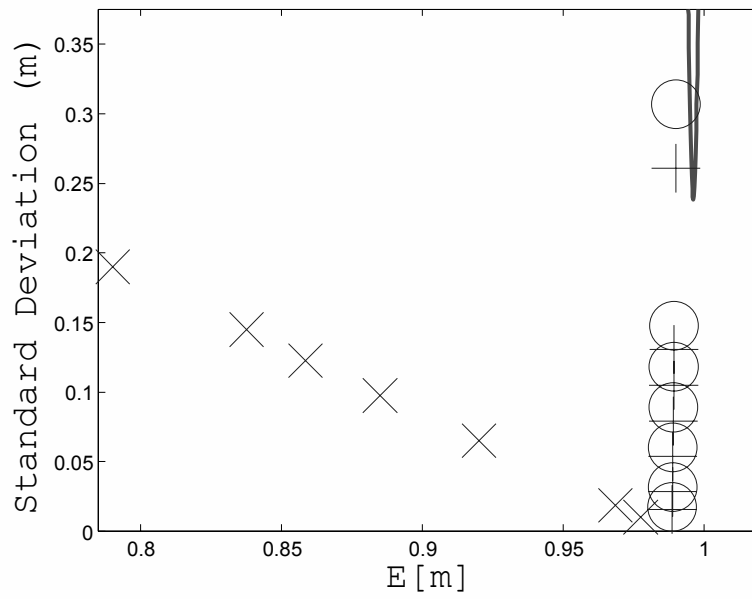
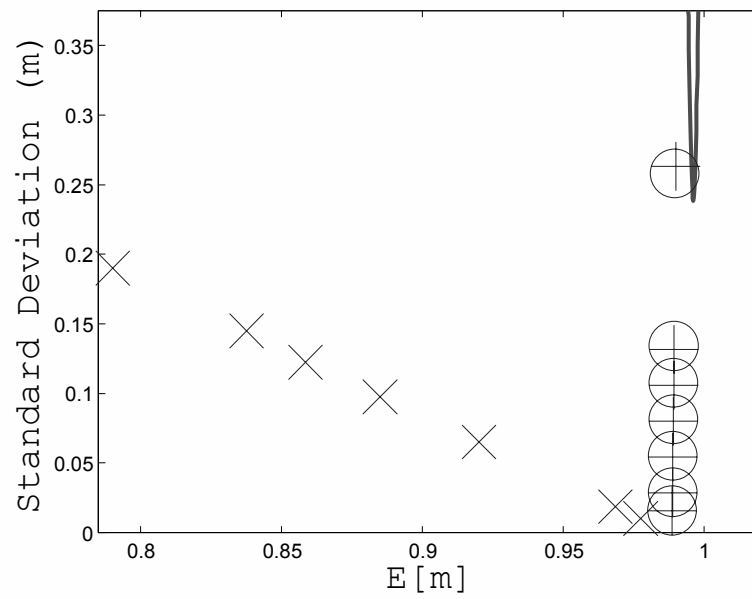
(a) $\overline{\sigma}_a = 0.0115$ for All Three Calibrations(b) $\overline{\sigma}_a$ Calibrated to Match $\Delta \log(c) 0.0055$

Fig. 4.6 Stochastic Volatility and the Hansen-Jagannathan Bounds
 \times : Expected Utility; $+$: Constant Volatility; \circ : Baseline (Stochastic Volatility)

ever, at the cost of increasing the variance of the log consumption growth. As figure 4.6b shows, if we hold that variance constant at its empirical counterpart by reducing the homoskedastic component of the productivity growth shock—as Tallarini (2000) does throughout his study, the effect of volatility shock in terms of further increasing the volatility in the pricing kernel vanishes, reiterating the conditional heteroskedastic interpretation of volatility shocks.

This casts doubt on the portability of the results of Bansal and Yaron (2004) and others that identify stochastic volatility as a potential contributor to the resolution of asset pricing puzzles summarized in the pricing kernel's ability to reach the Hansen-Jagannathan bounds. When an endowment setting is abandoned in favor of a production model, the variance of log consumption growth can no longer be held exogenously. Increasing the variance of the volatility process leads to an increase in the variance of the log consumption process. Holding the overall volatility constant by adjusting the homoskedastic component of the productivity process downward counteracts the increased variance of the volatility process nearly completely, leaving the model as well off with regards to the Hansen-Jagannathan bounds as without stochastic volatility.

4.8 An Extension to Frictional Investment

In this section, we extend the model in section 5.2 to demonstrate that when the model is no longer frictionless an increase in risk may lead to a fall in output as argued for in Bloom (2009), Fernández-Villaverde et al. (2011b), Basu and Bundick (2012), Bloom et al. (2012) and Bidder and Smith (2012). To accomplish this, we extend the model in two dimensions. First, we add variable capital utilization with endogenous depreciation to enable households to accumulate their precautionary buffer stock of capital in response to a volatility shock by reducing capital utilization and thereby decreasing depreciation as an alternative to increasing investment.³¹ Second, we impose capital adjustment costs to increase the relative attractiveness of this alternate channel of capital accumulation with respect to increasing investment.³² A numerical analysis of this extended model suggests, when capital adjustment cost is sufficiently high and thus household primarily chooses to decrease utilization rate to build up buffer stock of

³¹Variable capital utilization allows household to adjust capital in service immediately, as opposed to a time-to-build fashion, in response to shocks that alters the marginal productivity of capital, see Greenwood et al. (1988), Burnside and Eichenbaum (1996) and King and Rebelo (1999) for detailed analysis of models' propagation mechanism in presence of variable capital utilization.

³²See Hayashi (1982) for the theoretical foundation of capital adjustment cost, Jermann (1998) for its application in asset pricing and Baxter and Crucini (1993), Baxter and Crucini (1995) and Baxter and Farr (2005) for its contribution to explaining international trade and business cycles.

capital in response to a volatility shock, the resulting decrease in capital for production and the consequential fall in current output outweighs the simultaneous increase in output induced by increased labor input.

The infinitely lived household still seeks to maximize its expected discounted lifetime utility given by the recursive preferences (4.10) over consumption and labor supply subject to the budget constraint (4.12). The representative firm now maximizes profits $Y_t - W_t N_t - I_t$ in each period by choosing labor input, investment and the capital utilization rate, subject to the following capital accumulation law and production technology³³

$$Y_t = (u_t K_{t-1})^\alpha (e^{Z_t} N_t)^{1-\alpha} \quad (4.48)$$

$$K_t = (1 - \delta_t) K_{t-1} + \phi_t K_{t-1} \quad (4.49)$$

The capital adjustment cost function, ϕ_t in (4.49) penalizes investment, in units of current capital, for deviating from its frictionless level, and follows Jermann's (1998) specification

$$\phi_t = \phi \left(\frac{I_t}{K_{t-1}} \right) = \frac{b_k}{1 - 1/\xi_k} \left(\frac{I_t}{K_{t-1}} \right)^{1-1/\xi_k} + c_k \quad (4.50)$$

where b_k and c_k are constants that will be set to ensure that adjustment costs are neutral in the deterministic steady state and ξ_k the elasticity of the investment capital ratio with respect to Tobin's q .³⁴ With variable capital utilization, u_t in (4.49) and (4.48), firms can adjust the capital input in production contemporaneously. However, increasing u_t leads to faster capital depreciation and the depreciation function follows Baxter and Farr's (2005) specification

$$\delta_t = \delta(u_t) = \frac{b_u}{1 + \xi_u} u_t^{1+\xi_u} + c_u \quad (4.51)$$

where b_u and c_u are constants that will be chosen such that capital is fully utilized in the deterministic steady state and ξ_u the elasticity of marginal depreciation with respect to the utilization rate.³⁵ The model is closed by the market clearing condition (5.13) as before.

The firm's optimal utilization plan in presence of capital adjustment cost, equating the marginal benefit in terms of additional output produced to the marginal cost in

³³We follow Uzawa (1969) and introduce adjustment costs associated with investment. See, e.g., Lucas (1967) for an alternative.

³⁴I.e., $\xi_k \equiv -(\phi_t''/\phi_t')/(I_t/K_{t-1})$ where $\phi_t' = \partial \phi_t / \partial (I_t/K_{t-1})$ the marginal capital adjustment cost and the inverse of Tobin's q , and $\phi_t'' = \partial \phi_t' / \partial (I_t/K_{t-1})$, see Baxter and Crucini (1995) for example.

³⁵I.e., $\xi_u \equiv u_t \delta_t'' / \delta_t'$ where $\delta_t' = \partial \delta_t / \partial u_t$ the marginal capital utilization and $\delta_t'' = \partial \delta_t' / \partial u_t$, see Baxter and Farr (2005) for example.

terms of additional units of, is capital being worn out

$$\alpha \frac{Y_t}{u_t} = \frac{\delta'_t}{\phi'_t} K_{t-1} \quad (4.52)$$

The risky rate of return on capital in the presence of both capital adjustment cost and variable utilization is now

$$r_t = \left(\alpha \frac{Y_t}{K_{t-1}} + \frac{1 - \delta_t + \phi_t}{\phi'_t} - \frac{I_t}{K_{t-1}} \right) \phi'_{t-1} - 1 \quad (4.53)$$

We keep the parameters of the baseline model in section 5.2 at their values stated there (see Table 4.1), except for the homoskedastic component of the standard deviation of productivity growth, $\overline{\sigma}_z$, which we adjust to match the standard deviation of (log) consumption growth. As do Christiano et al. (2005) and Ríos-Rull et al. (2012), we impose full capital utilization in the deterministic steady state by letting $\bar{u} = 1$, and ensure the adjustment cost does not affect the deterministic steady state by setting $\bar{\phi} = \bar{I}/\bar{K}$ and $\bar{\phi}' = 1$ as also noted in van Binsbergen et al. (2012).

For ξ_k , the literature varies in the range from 0.101 from van Binsbergen et al. (2012), to 0.23 from Jermann (1998),³⁶ to 13.3 from Baxter and Jermann (1999) and to 15 from Baxter and Crucini (1995) with changes in investment becoming less costly as ξ_k increases (See Baxter and Crucini (1993)). We set $\xi_k = 1.5$, making utilization rate a preferred channel of adjusting capital in service in response to a volatility shock as changes in investment is fairly costly with this value. Note that, as the value of ξ_k increases and adjusting capital through investment becomes less costly, the extended model might again predict a boom in response to an increase in risk.

For the elasticity of utilization, ξ_u , Baxter and Farr (2005) examine three values: $\xi_u = 1$, taken from Basu and Kimball (1997),³⁷ 0.1 taken from King and Rebelo (1999), along with a highly elastic case under the value 0.05. We set $\xi_u = 0.1$ and note that the primary concern of the analysis of this extended model, the response of variables to a volatility shock, is qualitatively robust to this highly elastic case as well.

Figure 4.7 depicts the impulse responses of macroeconomic variables, expressed as deviations from their risk corrected steady states, under a third order approximation to a positive, one standard deviation volatility shock. As in the baseline model (See Figure 5.3), the household accumulates a buffer stock of capital in response to the increased volatility of future productivity shocks. To accumulate this stock, however, the household decreases the utilization rate, thereby slowing down depreciation. With this additional margin available to the household to accumulate capital, the increase in investment relative to capital, financed by decreasing consumption, is noticeably smaller

³⁶0.23 is near the lower bound of the empirical range as noted by Christiano et al. (2001)

³⁷Baxter and Farr (2005) note the imprecision of Basu and Kimball's (1997) estimation of this value.

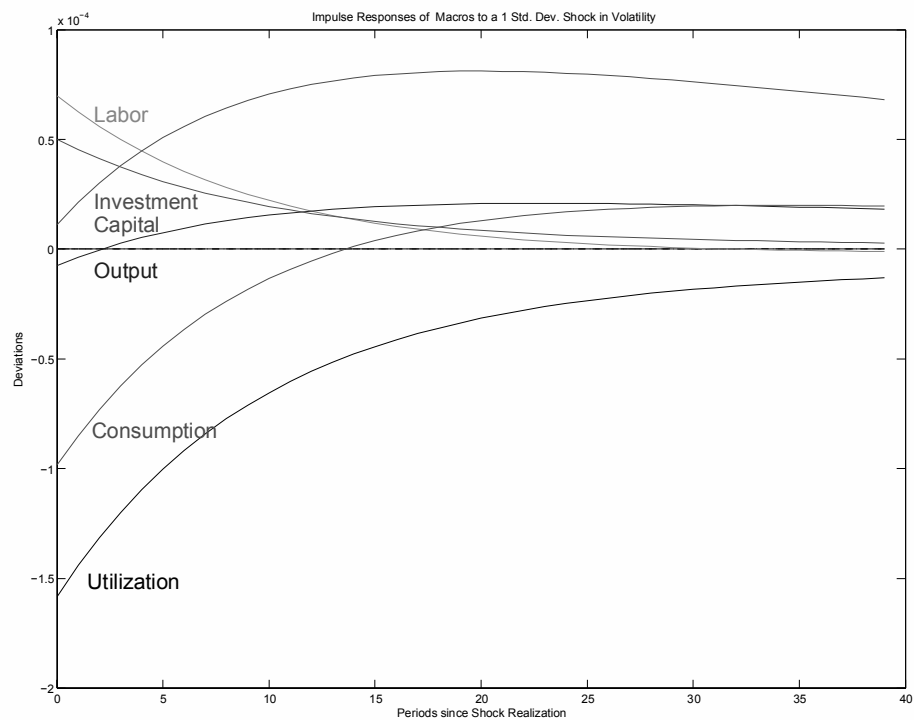


Fig. 4.7 Macro IRFs: Volatility Shock, Extended Model of Section 5.4

than that in the model with baseline calibration. In addition, the labor supply equation (4.17) still implies an increase in the marginal utility of consumption following the decrease in consumption to finance increased investment and leads to an increase in the marginal utility of leisure, and therefore a decrease in time spent on leisure. Unlike in the model under baseline calibration, the increased labor effort, with productivity having not changed (again, it is only the volatility of the distribution of future productivity shocks that is being shocked), fails to increase current output as the effect on output from the decrease in utilization rate and the consequential under-deployment of capital is stronger. Thus, this model predicts a fall in economic activity following an increase in risk as opposed to a boom predicted by the model with baseline calibration. The volatility shock is persistent but not permanent and as the shock dies out and productivity shocks fail to materialize from their widened distribution, the household winds down its buffer stock of capital by increasing consumption, leisure and the utilization rate, leading to a fall in investment but a rise in output as the effect of the increase in the utilization rate again dominates. Both the duration of the drop in output and its subsequent overshooting are consistent with the results documented by Bloom (2009).

Though, labor and investment still rise in response to a volatility shock here. Additions to the model, such as in Bidder and Smith (2012), with Jaimovich and Rebelo's (2009) preferences and Constantinides's (1990) consumption habit formation, show that a positive volatility shock leads to a simultaneous drop in output, investment, utilization, consumption and labor. Owing to Jaimovich and Rebelo's (2009) preferences, labor supply in their model is largely independent of wealth effects, and thus declines with other macroeconomic quantities. The habit formation slows down consumption adjustment, increasing the persistence and magnitude of a recession. By taking the demand side into account, Basu and Bundick (2012) show this uniform drop in macroeconomic aggregates in response to an increase in risk using a New Keynesian model with a countercyclical markup through sticky prices. On impact of a volatility shock, the increased labor supply as a precautionary reaction reduces firms marginal cost of production and thereby increases the markup since price is sticky. A higher markup winds down the demand for both consumption and investment goods, leading to a fall in output and employment. Nonetheless, it is noteworthy that adjustment costs and variable capital utilization alone are sufficient to generate the drop in output that Bloom (2009) identified empirically.

The third column of Table 4.6 reports the theoretical standard deviation of the extended model. Comparing to the model with baseline calibration, the standard deviation of the logarithmic investment capital ratio is noticeably smaller, consistent with

Table 4.6 Standard Deviation Comparison

Variable	Baseline Calibration	Extended Model	Data
$\Delta \log(c)$	0.0055	0.0055	0.0055
$\Delta \log(y)$	0.0096	0.0058	0.0104
$\Delta \log(i)$	0.0240	0.0068	0.0279
$\log(i) - \log(k)$	0.0707	0.0368	-
$\log(c) - \log(y)$	0.0154	0.0018	0.0377
$\log(i) - \log(y)$	0.0425	0.0052	0.0649

Table 4.7 Variance Decomposition in Percentage

	Constant Vol. Cal.		Baseline Cal.		Ext. Model	
	1st amp.	risk adj.	1st amp.	risk adj.	1st amp.	risk adj.
<i>cmpr</i>	0	100	0	100	0	100
<i>erp</i>	0	100	0	100	0	100
<i>rp</i>	109.03	0.60	80.68	8.51	79.45	0.95
<i>rf</i>	105.56	0.14	80.35	0.86	14.83	81.54
$\log(k)$	96.35	0.02	75.08	1.37	80.03	0.21
<i>i</i>	95.28	0.02	57.19	30.25	78.39	1.03
$\log(c)$	96.65	0.01	75.88	2.62	80.02	0.23
$\log(y)$	95.03	0.04	44.54	36.99	80.09	0.21
$\log(N)$	97.82	0.01	66.25	18.61	78.76	1.44

For each calibration, the columns may not add up to 100 due to the omission of 2nd and 3rd order amplification and cross effects.

the interpretation that adjusting capital through investment is very costly in the presence of adjustment cost, leaving investment relatively less volatile. The substantial drop in the standard deviation of the logarithmic investment output ratio reinforces this interpretation as the volatility of investment now contributes much less, through capital, to that of output when the utilization margin is activated.

The last two columns of Table 4.7 report the variance decomposition result of the extended model. Comparing to the results of the model with baseline calibration reported in the fourth and fifth columns, the contribution from the time varying risk adjustment channel to the total variation of all the listed variables, except conditional market price of risk and conditional risk premium, drops dramatically. This is not surprising, as the production side of the extended model is less risky than that of the model with baseline calibration — the presence of variable utilization and adjustment cost highlights the intratemporal substitution effects in response to shocks, i.e., household can adjust capital and thereby output immediately on the impact of shocks, as

noted by Greenwood et al. (1988), Burnside and Eichenbaum (1996) and King and Rebelo (1999), and need not wait till next period. This is in contrast to the baseline model, where the capital input could not be adjusted in response to shifts in risk (fixed utilization rate and time to build capital). Variable utilization and adjustment cost tend to increase the volatility of the pricing kernel as noted in Cochrane (2005),³⁸ by stretching out its time varying risk adjustment channel and thus shifting risk adjustments to risk free rate.

4.9 Conclusion

We have studied a business cycle model with recursive preferences and stochastic volatility with a third order perturbation approximation to the nonlinear moving average policy function. We use the impulse responses generated by this third order approximation to analyze the propagation mechanism of a volatility shock, and find that for macroeconomic variables, a volatility shock by itself propagates solely through a time-varying risk adjustment channel. For conditional asset pricing variables, this time-varying risk adjustment channel is the only working channel for the transmission of shocks, both to productivity growth and its volatility.

We have derived a closed-form calculation of the theoretical moments of the endogenous variables using a third order approximation. Our calculation of moments lends itself to a decomposition that disentangles the individual contributions of time-varying risk adjustment and amplification channels to the total variance. In our baseline model, we find that adding stochastic volatility alters the composition of variance, making a time-varying risk channel a substantial contributor of variance. For macroeconomic variables, variations that come from the time-varying risk adjustment channel can be explained by the household's precautionary savings desires and, in the presence of stochastic volatility, we find a large portion of variations in macroeconomic variables is driven by precautionary behavior.

Our extended model with frictional investment predicts a drop and subsequent overshooting of output in response to a volatility shock, consistent with empirical findings. Yet, with variable capital utilization, the capital input in production can be adjusted contemporaneously in response to shocks, eliminating the importance of the time-varying risk adjustment channel for macroeconomic variables. This finding, skeptical of the importance of stochastic volatility for precautionary behavior in production mod-

³⁸The standard deviation of risk free rate of the extend model is smaller than that of the model with baseline calibration, as variable utilization reduces the overall volatility of the extend model.

els, is corroborated by our finding that stochastic volatility contributes to the baseline model's ability to reach the Hansen-Jagannathan bounds only inasmuch as it increases the overall volatility of the model.

In linear approximations, variance decompositions can be applied to study the individual contribution of each shock to the total variance. The channels of risk adjustment and amplification we have derived here are a first step towards a shock-specific decomposition of nonlinear perturbation approximations. This would enable the identification of the individual contributions of each shock, not only to total volatility, but also to individual orders of nonlinearity and risk adjustments.

4.10 Acknowledgements

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Chapter 5

Stochastic Volatility in Real General Equilibrium

Hong Lan*

Abstract

In this paper I examine the propagation mechanism of stochastic volatility in a neoclassical growth model that incorporates labor market search, adjustment cost to investment, variable capital utilization and a weak short-run wealth effect, but no nominal frictions such as sticky wage and price. In this general equilibrium environment, stochastic volatility generates business cycle fluctuations in major macroeconomic aggregates due to the precautionary motive of risk-averse agents, yet it has no significant effects on these major aggregates as suggested by the numerical analysis of the model.

JEL classification: C63, C68, E32

Keywords: Stochastic volatility; DSGE; search and matching; nonlinear perturbation

*Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Email:lanhong@cms.hu-berlin.de

5.1 Introduction

The propagation mechanism and quantitative importance of stochastic volatility in general equilibrium is still an ongoing discussion. I derive a DSGE model with stochastic volatility embedded, incorporating no nominal frictions but only adjustment cost to investment, and thereby provide a general equilibrium environment that contains real frictions only to evaluate the qualitative and quantitative implications of stochastic volatility.

Often modeled as volatility shock, stochastic volatility generates business cycle fluctuations in macroeconomic aggregates by triggering the precautionary reactions of risk-averse households as it alters the distribution of future risk. In the baseline model where labor market search and matching à la Mortensen and Pissarides is embedded, a positive shock in the volatility of productivity increases the uncertainty in the realization of future productivity. In response to this increased risk, households lower current consumption owing to the precautionary motive, leading to an increase in the marginal utility of consumption. This increase causes the marginal cost of vacancy creation (marginal welfare loss due to vacancy creation from planner's perspective) in consumption terms to rise, and accordingly firms (planner) create less vacancies. The reduction in current vacancy then leads to a fall in future employment and output under conventional calibration. In the extended model that includes investment adjustment cost and variable capital utilization in addition to labor market search, the increased marginal utility of consumption also causes the value of current installed capital in consumption terms to rise, giving an incentive to households to slow down the depreciation of current capital stock by lowering the utilization rate, resulting in a fall in current effective capital and investment. In sum and with a weak, short-run wealth effect introduced by using the Jaimovich and Rebelo's (2009) preferences, output, consumption, investment, employment and capital in service in the extended model fall together in response to a positive shock in the volatility of productivity. The systematic reaction and positive comovement among these aggregates in responses to a positive shock in the volatility of investment specific technology shock, preferences shock and government spending shock can be likewise explained by the precautionary motive and the chain reaction it will initiate.

Alternative to the propagation mechanism proposed by Basu and Bundick (2012) which is based on sticky price and wage setting, neither the baseline nor the extended model includes such nominal rigidities. Moreover, as the Hosios's (1990) condition holds by construction, labor market search and matching frictions as a special type of labor adjustment cost can be internalized. The extended model therefore only con-

tains as frictions adjustment cost to investment. In this general equilibrium environment, I find that quantitatively, the impact of stochastic volatility on macroeconomic aggregates is minimal. Even if the size of volatility shocks are reasonably large, the responses to these shocks are very small. In addition, while stochastic volatility significantly enlarges the conditional standard deviation of the aggregates, its contribution to the unconditional standard deviation is small. This result is in agreement with those reported by Bachmann and Bayer (2013), Bachmann et al. (2013) and Born and Pfeifer (2014). Furthermore, it provides a potential explanation to the sizable impact of stochastic volatility reported by Fernández-Villaverde et al. (2011a) — as they study stochastic volatility in a monetary, general equilibrium model in which a certain amount of nominal rigidities are embedded, it is reasonable to argue that the substantial effect of stochastic volatility they have observed may depend on the presence of the nominal rigidities.

The paper is organized as follows. The baseline model is presented in section 5.2. In section 5.3, I briefly introduce the nonlinear moving average perturbation that used to solve the model, and lay out the baseline calibration for numerical analysis of the model. I present the impulse responses and moments of the baseline model and explain the propagation mechanism in section 5.4. In section 5.5, the baseline model is extended to incorporate investment adjustment cost and variable capital utilization, together with some other features that are frequently modeled in the study of stochastic volatility. The propagation mechanism is reexamined with the presence of those new ingredients. Section 5.6 concludes.

5.2 The Baseline Model

In this section, I derive a neoclassical growth model with stochastic volatility in productivity. The labor market in this model is characterized by search and matching frictions à la Mortensen and Pissarides, implemented as in Merz (1995) and Andolfatto (1996).

5.2.1 The Baseline Model

The economy is populated by infinitely lived, identical households with Jaimovich and Rebelo's (2009) preferences (hereafter JR preferences)

$$U_t = \frac{\left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t\right)^{1-\kappa_F} - 1}{1 - \kappa_F} \quad (5.1)$$

with

$$S_t = c_t^{\kappa_W} S_{t-1}^{1-\kappa_W} \quad (5.2)$$

where c_t is consumption, n_t the fraction of employed family members. The presence of S_t makes preferences non-time-separable in c_t and n_t , and $\kappa_W \in [0, 1]$ governs the size of wealth effect. κ_N is a strictly positive constant that scales the size of disutility rising from work, κ_F the risk aversion parameter and γ the inverse of the Frisch elasticity of labor supply. When $\kappa_W = 1$, the JR preferences (5.1) turn into the preferences discussed in King et al. (1988) (hereafter KPR preferences), and when $\kappa_W = 0$, it amounts to the class of preferences proposed by Greenwood et al. (1988) (hereafter GHH preferences).

Households own the capital in the economy, and maximize the present discounted value of their life-time utility by choosing capital investment

$$\max_{i_t} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U_t \quad (5.3)$$

subject to (5.2) and the following budget constraint

$$c_t + i_t = w_t n_t + r_t k_t \quad (5.4)$$

where i_t is investment, $\beta \in (0, 1)$ the discount rate, w_t wage and r_t the rental rate of capital. Households accumulate capital according the following law of motion

$$k_{t+1} = (1 - \delta)k_t + i_t \quad (5.5)$$

where $\delta \in (0, 1)$ is the capital depreciation rate. Similarly, the aggregate employment evolves according to the following

$$n_{t+1} = (1 - \chi)n_t + m_t \quad (5.6)$$

where $\chi \in (0, 1)$ denotes the exogenous constant job separation rate. m_t represents the number of job matches that are created in period t . Following Merz (1995), Ando-fatto (1996), Pissarides (2000), Shimer (2005), Pissarides (2009) and many others, job matches are assumed to be generated by a Cobb-Douglas matching function

$$m_t = m_0 v_t^{1-\eta} (1 - n_t)^\eta \quad (5.7)$$

where m_0 is a constant scaling factor and $\eta \in (0, 1)$ the elasticity of the matching function with respect to unemployment. The aggregate employment in the next period

therefore is the sum of current employment that has not been destroyed, and the new employment generated by the matching function.

Competitive firms choose the amount of capital to rent from households and the number of vacancies to create in order to maximize the sum of their discounted, expected profit

$$\max_{k_t, v_t} \mathbb{E}_t \sum_{t=0}^{\infty} \left(\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \right)^t (y_t - w_t n_t - r_t k_t - \kappa_V v_t) \quad (5.8)$$

subject to

$$n_{t+1} = (1 - \chi) n_t + q_t v_t \quad (5.9)$$

where $\lambda_{1,t}$ denotes the marginal utility of consumption defined in section 5.2.2, κ_V the constant vacancy posting cost and q_t vacancy filling rate that measures the rate at which vacancies become filled. Firms employ a labor-augmenting production function in Cobb-Douglas form to produce output y_t in period t

$$y_t = k_t^\alpha (e^{z_t} n_t)^{1-\alpha} \quad (5.10)$$

where $\alpha \in (0, 1)$ is the capital share in production and z_t the productivity level that follows

$$z_t = \rho_z z_{t-1} + e^{\sigma_{z,t}} \varepsilon_{z,t}, \quad \varepsilon_{z,t} \sim \mathcal{N}(0, 1) \quad (5.11)$$

where ρ_z is the persistence parameter and $\varepsilon_{z,t}$ the productivity shock. As in Fernández-Villaverde et al. (2011b) and Caldara et al. (2012), $\varepsilon_{z,t}$ is scaled by a stochastic volatility level $\sigma_{z,t}$, which evolves as follows

$$\sigma_{z,t} = (1 - \rho_{\sigma_z}) \overline{\sigma_z} + \rho_{\sigma_z} \sigma_{z,t-1} + \tau_z \omega_{z,t}, \quad \omega_{z,t} \sim \mathcal{N}(0, 1) \quad (5.12)$$

where $\overline{\sigma_z}$ is the unconditional mean level of $\sigma_{z,t}$, ρ_{σ_z} the persistence parameter and $\omega_{z,t}$ the innovation in $\sigma_{z,t}$ that is scaled by a constant τ_z . The model is closed by the following resource constraint

$$c_t = y_t - i_t - \kappa_V v_t \quad (5.13)$$

By assuming that households and firms share the job match surplus according to firms' recruiting effort $1 - \eta$, there are no search and matching externalities. The model is thereby frictionless and can be presented as a social planner's problem¹

$$V(k_t, n_t, z_t, \sigma_{z,t}) = \max_{c_t, v_t} \left\{ U_t + \beta \mathbb{E}_t V(k_{t+1}, n_{t+1}, z_{t+1}, \sigma_{z,t+1}) \right\} \quad (5.14)$$

subject to (5.1), (5.2), (5.6), (5.7), (5.10), (5.11), (5.12) and the following resource

¹ Except for the time-varying volatility of productivity and the JR preferences, the baseline model is a special case of the stationary version of Merz (1995) with zero search cost, and therefore her proof of the equivalence between the market model and the planner's problem directly applies.

constraint

$$k_{t+1} = (1 - \delta)k_t + y_t - c_t - \kappa_v v_t \quad (5.15)$$

which states the capital stock in the next period as the sum of current capital after depreciation and current output, net of consumption and the total cost of vacancy posting. Moreover, since the model assumes only two states for a family member, employed or unemployed, the fraction of the unemployed family members writes

$$u_t = 1 - n_t \quad (5.16)$$

As is usual in labor market search and matching literature, the vacancy filling rate q_t , job finding rate f_t and labor market tightness θ_t are defined as follows

$$q_t \equiv \frac{m_t}{v_t} \quad (5.17)$$

$$f_t \equiv \frac{m_t}{1 - n_t} = \frac{m_t}{u_t} \quad (5.18)$$

$$\theta_t \equiv \frac{v_t}{1 - n_t} = \frac{v_t}{u_t} \quad (5.19)$$

Both the job finding and vacancy filling rate are probabilities, and should lie between zero and one. The vacancy filling rate, however, can potentially exceed unity in simulation when the matching function takes the Cobb-Douglas form (see den Haan et al. (2000, p. 485)). To avoid introducing nonsmoothness into the policy function since in that case the perturbation methods cannot be applied, I do not restrict q_t to be less than one. The realization of q_t that exceeds unity is interpreted as that firms hire more than one worker on each posted vacancy, see Den Haan and De Wind (2012a).

5.2.2 Characterization

Apart from the constraints, the social planner's optimization problem is characterized by the following set of first order necessary conditions

$$\lambda_{1,t} = \left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t \right)^{-\kappa_F} + \lambda_{2,t} \kappa_W c_t^{\kappa_W-1} S_{t-1}^{1-\kappa_W} \quad (5.20)$$

$$\lambda_{2,t} = -\kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} \left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t \right)^{-\kappa_F} + \beta(1 - \kappa_W) \mathbb{E}_t (\lambda_{2,t+1} c_{t+1}^{\kappa_W} S_t^{-\kappa_W}) \quad (5.21)$$

$$\lambda_{3,t} = \lambda_{1,t} \frac{\kappa_V}{m_{v,t}} \quad (5.22)$$

$$\lambda_{1,t} = \beta \mathbb{E}_t [\lambda_{1,t+1} (1 - \delta + y_{k,t+1})] \quad (5.23)$$

$$\lambda_{3,t} = \beta \mathbb{E}_t [\lambda_{1,t+1} y_{n,t+1} + U_{n,t+1} + \lambda_{3,t+1} (1 - \chi + m_{n,t+1})] \quad (5.24)$$

where $\lambda_{1,t}$, $\lambda_{2,t}$ and $\lambda_{3,t}$ are the Lagrange multipliers associated with (5.15), (5.2) and (5.6) respectively. Given the production function (5.10), the marginal productivity of

capital and labor writes

$$y_{k,t} = \alpha k_t^{\alpha-1} (e^{z_t} n_t)^{1-\alpha} \quad (5.25)$$

$$y_{n,t} = (1-\alpha) k_t^\alpha (e^{z_t})^{1-\alpha} n_t^{-\alpha} \quad (5.26)$$

Given the utility function (5.1) and the matching function (5.7), the disutility from work and the marginal contribution from vacancy and employment to job matches writes

$$U_{n,t} = -\kappa_N s_t n_t^\gamma \left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t \right)^{-\kappa_F} \quad (5.27)$$

$$m_{v,t} = (1-\eta) m_0 v_t^{-\eta} (1-n_t)^\eta \quad (5.28)$$

$$m_{n,t} = -\eta m_0 v_t^{1-\eta} (1-n_t)^{\eta-1} \quad (5.29)$$

In this set of first order conditions, (5.20) denotes the marginal utility of consumption. (5.21) characterizes the dynamics of S_t in the JR preferences. From the planner's perspective, the Lagrange multiplier $\lambda_{3,t}$ in (5.22) represents the marginal welfare loss due to current vacancy creation, measured in consumption terms. Euler equation for consumption (5.23) equalizes the expected present-discounted utility value of postponing consumption of one period to its utility value today. Euler equation for employment (5.24) equalizes the marginal welfare loss induced by vacancy creation to its expected present-discounted marginal welfare gain. This gain is the sum of the marginal labor productivity, net of the disutility from work, and the its potential continuation. $m_{n,t+1}$ corrects the continuation as the future (un)employment stock has been changed by current vacancy creation.

5.3 Solution Method and Baseline Calibration

The baseline model described in section 5.2 does not have a known closed form solution and needs to be approximated with numerical methods. This section first introduces the method that will be used to approximate the solution, and then presents the baseline calibration for the numerical analysis of the model.

5.3.1 Perturbation Solution

As shown by Caldara et al. (2012) and Lan (2014), perturbation methods can solve such a model quickly with a degree of accuracy comparable to global methods. I use the nonlinear moving average perturbation derived in Lan and Meyer-Gohde (2013d) as it delivers stable nonlinear impulse responses and simulations and, as shown in Lan and Meyer-Gohde (2013c), enables analytical calculation of moments. The model is

solved to third order as at least a third order approximation is necessary for the analysis of the effect of stochastic volatility.

For the implementation of the nonlinear moving average perturbation, I collect the equilibrium conditions, i.e., the constraints of the social planner's problem with the two Euler equations, into a vector of functions

$$0 = \mathbb{E}_t[f(\mathcal{Y}_{t+1}, \mathcal{Y}_t, \mathcal{Y}_{t-1}, \varepsilon_t)] \quad (5.30)$$

where \mathcal{Y}_t is the vector of the endogenous variables, and ε_t the vector of the exogenous shocks, assuming the function f in (5.30) is sufficiently smooth and all the moments of ε_t exist and finite.

The solution to (5.30) is a time-invariant function \mathcal{Y} , taking as its state variable basis the infinite sequence of realized shocks, past and present, and indexed by the perturbation parameter $\sigma \in [0, 1]$ scaling the distribution of future shocks

$$\mathcal{Y}_t = \mathcal{Y}(\sigma, \varepsilon_t, \varepsilon_{t-1}, \dots) \quad (5.31)$$

Assuming normality of all the shocks and setting $\sigma = 1$ as I am interested in the stochastic model, the third order approximation—a Volterra expansion, see Lan and Meyer-Gohde (2013a)—of (5.31), takes the form

$$\begin{aligned} \mathcal{Y}_t^{(3)} = & \overline{\mathcal{Y}} + \frac{1}{2} \mathcal{Y}_{\sigma^2} + \frac{1}{2} \sum_{i=0}^{\infty} (\mathcal{Y}_i + \mathcal{Y}_{\sigma^2, i}) \varepsilon_{t-i} + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \mathcal{Y}_{j, i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \\ & + \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \mathcal{Y}_{k, j, i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i}) \end{aligned} \quad (5.32)$$

where $\overline{\mathcal{Y}}$ denotes the deterministic steady state of the model, at which all the partial derivatives $\mathcal{Y}_{\sigma^2}, \mathcal{Y}_{\sigma^2, i}, \mathcal{Y}_i, \mathcal{Y}_{j, i}$ and $\mathcal{Y}_{k, j, i}$ are evaluated. (5.32) is naturally decomposed into order of nonlinearity and risk adjustment— $\mathcal{Y}_i, \mathcal{Y}_{j, i}$ and $\mathcal{Y}_{k, j, i}$ capture the amplification effects of the realized shocks $(\varepsilon_t, \varepsilon_{t-1}, \dots)$ in the policy function (5.31) at first, second and third order respectively. The two partial derivatives with respect to σ , \mathcal{Y}_{σ^2} and $\mathcal{Y}_{\sigma^2, i}$ adjust the approximation for future risk.² While \mathcal{Y}_{σ^2} is a constant adjustment for risk and a linear function of the variance of future shocks³, $\mathcal{Y}_{\sigma^2, i}$ varies over time, interacting the linear response to realized shocks with the variance of future shocks essentially adjusting the model for time variation in the conditional volatility of future risk.

²More generally, a constant term, \mathcal{Y}_{σ^3} , at third order adjusts (5.32) for the skewness of the shocks. See Andreasen (2012b). As I assume all the shocks are normally distributed, \mathcal{Y}_{σ^3} is zero and not included in (5.32) and the rest of the analysis.

³See, Lan and Meyer-Gohde (2013a) for the derivation of this term.

5.3.2 Baseline Calibration

The baseline model is quarterly calibrated. Table 5.1 summarizes the parameter values.

Table 5.1 Baseline Calibration

Symbol	Description	Value	Source
β	Discount rate	0.99	Standard value
α	Capital share	0.34	Ríos-Rull et al. (2012)
δ	Capital depreciation rate	0.019	Ríos-Rull et al. (2012)
γ	Inverse of Frisch elasticity	1/0.72	Ríos-Rull et al. (2012)
κ_F	Risk aversion	2	Standard value
χ	Job separation rate	0.07	Merz (1995)
η	Matching elasticity w.r.t. unemployment	0.5	Pissarides (2009)
n_{ss}	Steady state employment	0.94	Pissarides (2009)
θ_{ss}	Steady state labor market tightness	0.72	Shimer (2005)
m_0	Job match scaling factor	0.36	Deduced
ρ_z	Persistence of productivity process	0.95	Caldara et al. (2012)
ρ_{σ_z}	Persistence of volatility shock	0.90	Caldara et al. (2012)
$\bar{\sigma}_z$	Unconditional mean of productivity shock	$\ln(0.007)$	Caldara et al. (2012)
τ_z	Standard deviation of volatility shock	0.06	Caldara et al. (2012)
κ_W	Wealth effect scaling factor	0.001	Enforcing GHH preferences
κ_V	Vacancy posting cost	0.257	Chosen to match $\sigma(\theta)/\sigma(p) = 7.56$
κ_N	Disutility scaling factor	0.888	Deduced

For the value of the Frisch elasticity, Ríos-Rull et al. (2012) argue that 0.72 and 1 are the most credible ones, whereas higher value can also be found in the literature, e.g., 1.25 from Merz (1995). I use 0.72 as the benchmark and will examine the quantitative implications of the model with higher Frisch elasticity. Likewise, I set the risk aversion parameter κ_F to 2 and will evaluate the effect of higher/lower risk aversion on the numerical performance of the model.

For the parameters of the stochastic volatility process, I follow Caldara et al. (2012) and set $\rho_{\sigma_z} = 0.90$ and $\tau_z = 0.06$ respectively, to match “the persistence and standard deviation of heteroskedastic component of the Solow residual during the last five decades.”

In particular, I set the size of wealth effect $\kappa_W = 0.001$ as in Jaimovich and Rebelo (2009), effectively imposing the GHH preferences. As preferences play a key role in shaping the dynamics of the baseline model, I will then analyze in detail the model with the KPR preferences.

Finally, I set the vacancy posting cost κ_V to 0.256, to match the empirical volatility of labor market tightness relative to that of labor productivity which is 7.56 as reported by Pissarides (2009).

5.4 Analysis of the Baseline Model

This section presents the impulse responses and theoretical moments of the baseline model. Analyzing these numerical implications leads to two observations. First, labor market search and matching, when combined with the class of preferences with little wealth effect, can generate positive comovement among consumption, output and employment in response to a shock in the volatility of productivity. Second, the impact of such a shock on major macroeconomic aggregates is quantitatively insignificant. Under the baseline calibration, output deviates from its third order accurate stochastic steady state by about -1.2×10^{-6} in response to a positive, one standard deviation shock in the volatility of productivity. Moments analysis also supports this observation by showing that the contribution from stochastic volatility to the unconditional volatility of macroeconomic aggregates is minimal.

5.4.1 Impulse Response

This section presents the impulse responses of major macroeconomic aggregates to a positive shock in the volatility of productivity, i.e., in $\omega_{z,t}$, then analyzes the role of several parameters and the preferences in shaping the responses.

Figure 5.1 depicts the impulse response and its contributing components for capital to a positive, one standard deviation shock in volatility of productivity. In both Figure 5.1 and 5.2, the upper panel displays the impulse responses at first, second and third order as deviations from their respective (non)stochastic steady states (themselves in the middle right panel). In the the middle left panel and the middle column of panels in the lower half of the figure, the contributions to the total impulse responses from the first, second and third order amplification channels, that is, \mathcal{Y}_i , $\mathcal{Y}_{i,i}$ and $\mathcal{Y}_{i,i,i}$ in the third order approximation (5.32), are displayed. Notice that there is no response in these amplification channels. All responses to this volatility shock come from the lower left panel of the figure where the time-varying risk adjustment channel $\mathcal{Y}_{\sigma^2,i}$ is displayed. In other words, for capital, a volatility shock by itself propagates solely through the time-varying risk adjustment channel.

Capital responds positively to this positive volatility shock. This captures the plan-

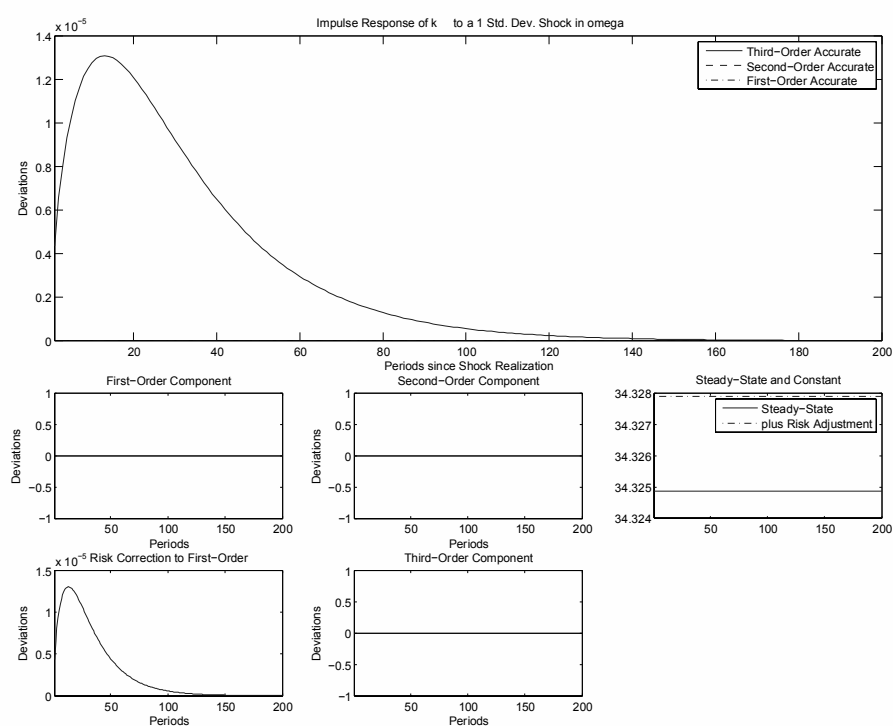


Fig. 5.1 Capital IRF: Volatility Shock, Baseline Model of Section 5.2

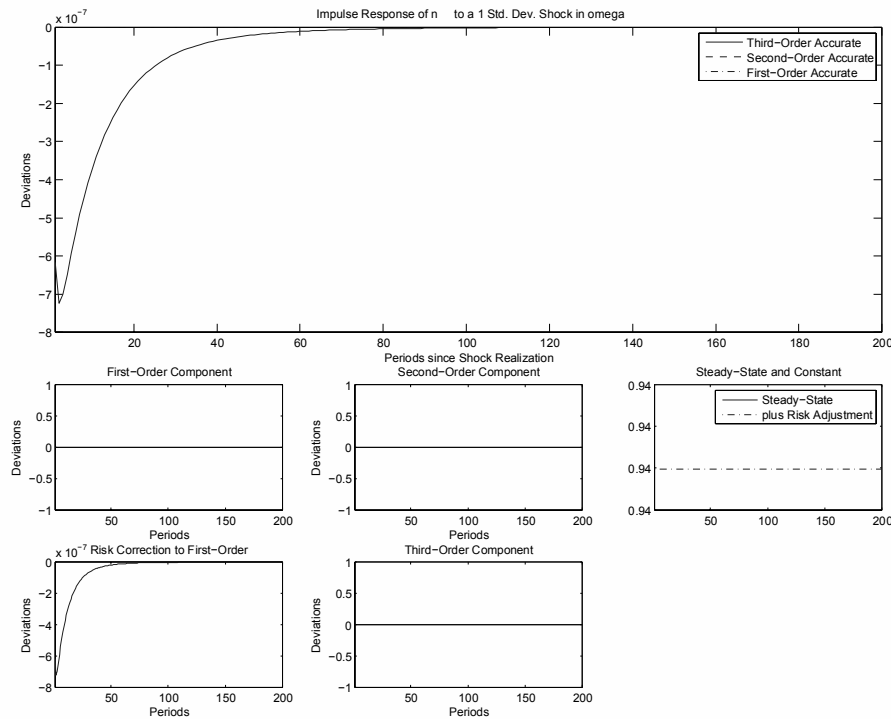


Fig. 5.2 Employment IRF: Volatility Shock, Baseline Model of Section 5.2

ner's precautionary reaction to the widening of the distribution of future productivity shocks.⁴ The risk-averse planner accumulates a buffer stock in capital to insure itself against the increased risk in future productivity as it will be drawn from a more dispersed distribution.

Figure 5.2 details the impulse response and its contributing components for employment to a positive, one standard deviation volatility shock to productivity. Like for capital, all responses of employment to this volatility shock comes from the time-varying risk adjustment channel and there is no response in any amplification channels. In the baseline model where employment is created by matching unemployed workers with vacancies, the negative response of employment is a direct consequence of the negative response of vacancy to a positive volatility shock to productivity, see Figure 5.3 below

Figure 5.3 displays the responses of consumption, investment, vacancy and output

⁴See also Fernández-Villaverde and Rubio-Ramírez (2010a) and van Binsbergen et al. (2012) for precautionary savings behavior in DSGE perturbation.

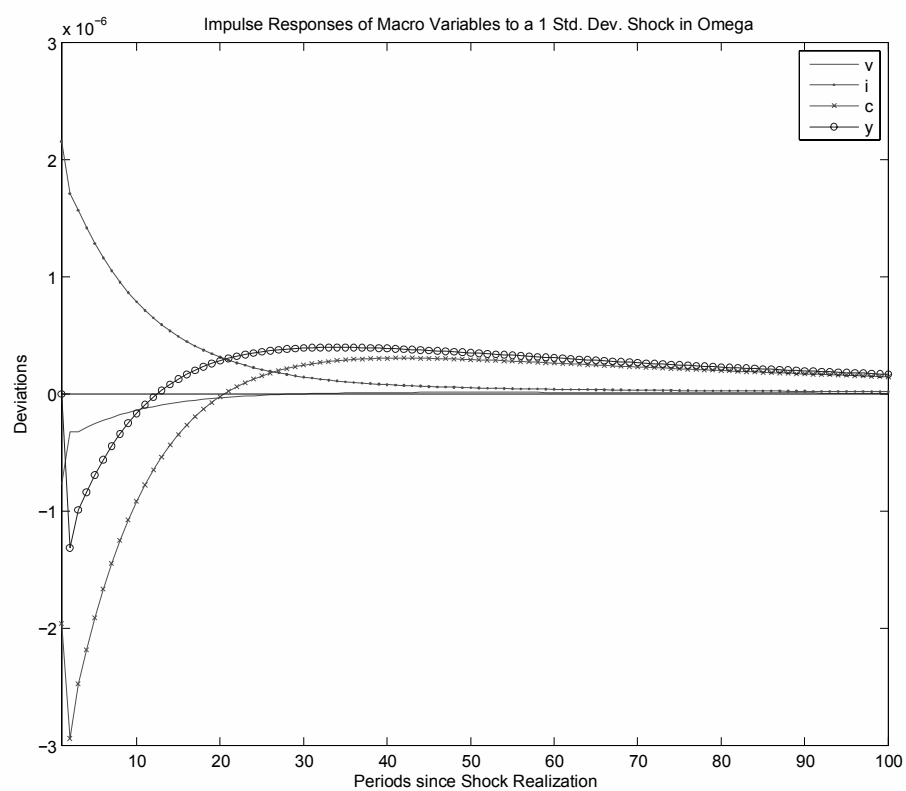


Fig. 5.3 Macro IRFs: Volatility Shock, Baseline Model of Section 5.2

as deviations from their third order accurate stochastic steady states to a positive, one standard shock in productivity. The social planner accumulates a buffer stock of capital by increasing current investment on impact of the shock. As the allocation has not changed, it finances this investment through a decrease in current consumption. With the capital stock being fixed on impact as it is a state variable and with the productivity having not changed,⁵ current output does not change on impact. The instantaneous increase in investment translate into an increase in capital stock in the next period according to the law of motion of capital (5.5). Furthermore, the decrease in current consumption results in an increase in the marginal utility of consumption, which in turn increases the marginal loss, in consumption terms, in welfare due to vacancy creation, see (5.22). Given the matching function (5.7) and that employment is a state variable that can not be adjusted on impact, the planner chooses to decrease its vacancy creation effort to counteract such additional welfare loss. As a result, less job match is created in current period (not pictured), translating into a drop in employment in the subsequent period according to the law of motion of employment (5.6).

Under the baseline calibration in section 5.3.2, the boosting effect from this increased capital stock on output is outweighed by the adverse effect from the decreased employment in the next period, resulting in a fall in output immediately after impact. Thus, the baseline model predicts a recession following an increase in risk of future productivity.⁶ The volatility shock is persistent but not permanent. As the shock dies out and productivity shocks fail to materialize from their widened distribution, the planner winds down its buffer stock of capital by increasing consumption and vacancy creation, leading to a fall in investment, an increase in employment and a quick rebound followed by an overshoot in output.⁷

Role of Risk Aversion, Frisch Elasticity and Job Separation Rate

To examine the role of risk aversion κ_F , the Frisch elasticity $1/\gamma$ and the job separation rate χ in shaping impulse responses, it is convenient to consider the baseline model with the exact GHH preferences, i.e., $\kappa_W = 0$. In this case S_t becomes a constant and

⁵Note that, it is the distribution governing future productivity shocks that is being shocked here, not the level of productivity itself.

⁶ While the impulse responses for the macroeconomic variables are not pictured with their contributing components, responses of these variables to a volatility shock come solely from the time-varying risk adjustment channel.

⁷This pattern of response of output to a positive volatility shock is consistent with that found by Bloom (2009).

can be normalized to one⁸, and the marginal utility of consumptions writes

$$\lambda_{1,t}^{GHH} = \frac{1}{\left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma}\right)^{\kappa_F}} \quad (5.33)$$

As shown in the preceding analysis of impulse responses, a positive shock to the volatility of productivity leads to an increase in the marginal utility of consumption. Note that $\frac{n_t^{1+\gamma}}{1+\gamma}$ is increasing in $1/\gamma$ given that $n_t \in (0, 1)$. Holding everything else constant, a fixed amount of drop in current consumption translates into a larger increase in $\lambda_{1,t}$ when the Frisch elasticity is high, and therefore a deeper cutback in vacancy than that with a lower Frisch elasticity. Consequentially, employment in the next period is lower, leading to a deeper contraction in output. See Figure 5.4 for the responses of consumption, the marginal utility of consumption and output to a positive, one standard deviation shock in volatility of productivity with $1/\gamma$ equals to 0.5 and 0.72 and 1.25 respectively.

The risk aversion parameter κ_F determines the magnitude of planner's precautionary motive. A highly risk-averse planner is motivated to build up a buffer stock of capital larger than that a less risk-averse planner would build though increasing current investment in response to an increase in future risk of productivity. When κ_F is extremely high, increasing current investment and cutting down current consumption is not enough to support the construction of the desirable amount of capital buffer stock. The planner then chooses to further decrease vacancy creation so that more resource can be used for investment. This leads to a deeper drop in employment, and therefore in output in the next period. See Figure 5.5 for the responses of consumption, investment, vacancy and output to a positive, one standard deviation shock in volatility of productivity with κ_F equals to 2, 10 and 20.

The job separation rate does not play a significant role in determining the response of the marginal utility of consumption to a volatility shock. Yet it can alter the size of the response of vacancy — as the law of motion of employment (5.6) implies, to reach the same amount of employment stock in the next period, the planner facing a high job separation rate needs to create more vacancies in current period to produce a larger employment inflow than that with a low job separation rate. Therefore, in response to a positive shock in volatility of productivity, the planner facing a low job separation rate needs to decrease vacancy further than that with a high job separation rate, leading to a lower employment stock in the next period and therefore a deeper drop in output.

Figure 5.6 depicts the responses of vacancy, employment and output to a positive,

⁸See Jaimovich and Rebelo (2009) for more details.

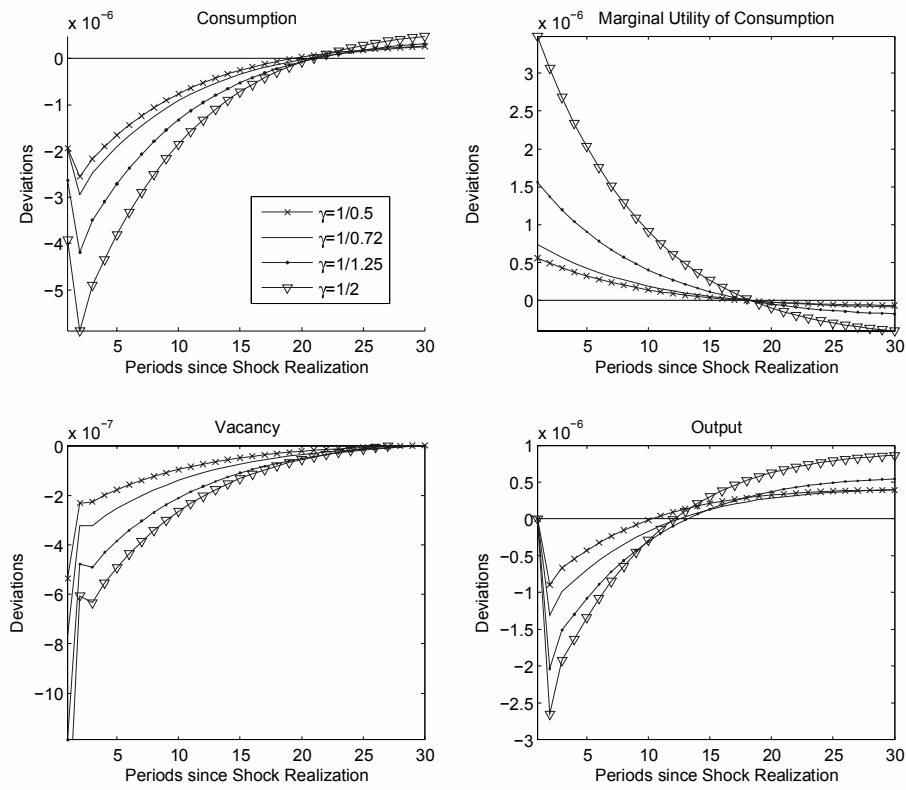


Fig. 5.4 Role of Frisch Elasticity

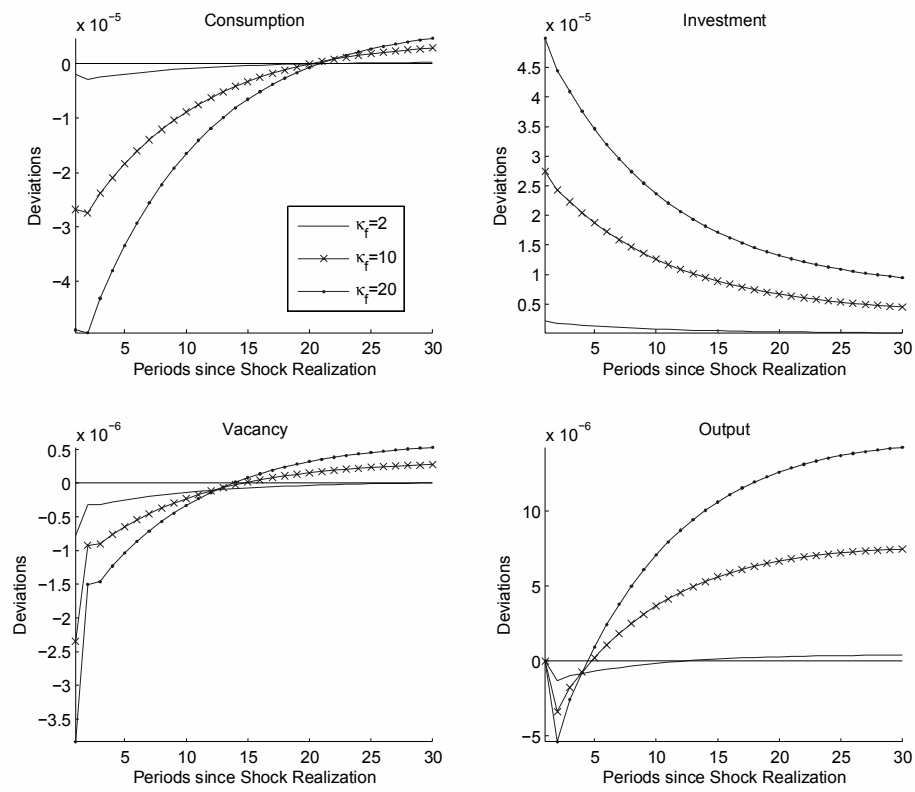


Fig. 5.5 Role of Risk Aversion

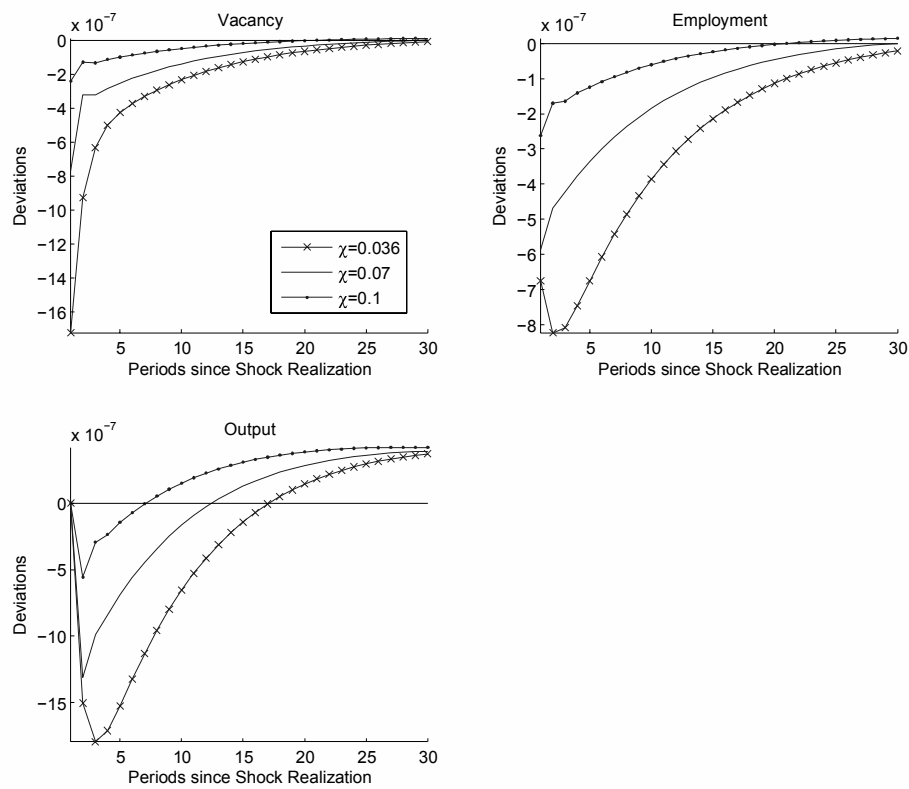


Fig. 5.6 Role of Job Separation Rate

one standard deviation volatility shock in productivity with χ equals to 0.1, 0.07 and 0.036 respectively. $\chi = 0.07$ is used in the baseline calibration and is taken from Merz (1995). $\chi = 0.036$ is the monthly separation rate reported by Shimer (2005) and Pissarides (2009) uses this value for quarterly calibration, assuming separation rate is constant within the quarter. Otherwise it aggregates to a quarterly separation rate of 0.1, see Shimer (2005).

The KPR Preferences and Volatility of Wage

When the baseline model is equipped with the KPR preferences, i.e., $\kappa_W = 1$, a positive shock in the volatility of productivity might lead to an increase in output. To understand the reason for such a counterintuitive result, it is useful to analyze the propagation mechanism of such a volatility shock in the market setup of the baseline model. In the market setup, firms' recruiting effort is characterized by the following first order necessary conditions

$$\lambda_{3,t} = \lambda_{1,t} \frac{\kappa_V}{q_t} \quad (5.34)$$

$$\lambda_{3,t} = \beta \mathbb{E}_t \left[\lambda_{1,t+1} \left(y_{n,t+1} - w_{t+1} + \frac{\kappa_V}{q_{t+1}} (1 - \chi) \right) \right] \quad (5.35)$$

where $\lambda_{3,t}$ in (5.34) is the marginal vacancy posting cost measured in consumption terms, and conditional on the current vacancy filling rate q_t . (5.35) equalizes that cost to its expected, discounted benefit. w_t in (5.35) denotes the market wage. Under the assumption that households and firms split match surplus according to firms' recruiting effort, the market wage takes the following form

$$w_t = \eta \left(y_{n,t} + \kappa_V \frac{v_t}{1 - n_t} \right) + (1 - \eta) \left(\frac{-U_{n,t}}{\lambda_{1,t}} \right) \quad (5.36)$$

With GHH preferences, the disutility of work writes

$$U_{n,t}^{GHH} = -\kappa_N n_t^\gamma \left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} \right)^{-\kappa_F} \quad (5.37)$$

Inserting the previous equation and the marginal utility of consumption (5.33) in (5.36) yields the market wage with the GHH preferences

$$w_t^{GHH} = \eta \left(y_{n,t} + \kappa_V \frac{v_t}{1 - n_t} \right) + (1 - \eta) \kappa_N n_t^\gamma \quad (5.38)$$

With KPR preferences, the marginal utility of consumption and disutility of work writes

$$\lambda_{1,t}^{KPR} = c_t^{-\kappa_F} \left(1 - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} \right)^{1-\kappa_F} \quad (5.39)$$

$$U_{n,t}^{KPR} = -\kappa_N n_t^\gamma c_t^{1-\kappa_F} \left(1 - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} \right)^{-\kappa_F} \quad (5.40)$$

Inserting the previous two equations in (5.36) yields the market wage with the KPR preferences

$$w_t^{KPR} = \eta \left(y_{n,t} + \kappa_V \frac{v_t}{1-n_t} \right) + (1-\eta) \kappa_N n_t^\gamma \left(\frac{c_t}{1 - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma}} \right) \quad (5.41)$$

The crucial difference between the two wages above is that w_t^{KPR} includes current consumption whereas w_t^{GHH} does not. In the light of Greenwood et al.'s (1988) interpretation, w_t^{GHH} is determined independently of households' intertemporal consumption decision, though such a wage can be considered as the result from a two-sided (households and firms) bargaining process. This property of w_t^{GHH} also enables the following interpretation of the propagation mechanism of a positive volatility shock in productivity — with the GHH preferences, firms can lower down current wage by creating less vacancies to insure themselves against the potential decrease in current profit in response to a positive volatility shock in productivity. On the other hand, households reduce current consumption to build up a buffer stock of capital. While this would increase the marginal utility of consumption and therefore the marginal, conditional cost of vacancy posting in consumption terms, such an increase has been offset by the decrease in firms' vacancy creation behavior which leads to a lower vacancy filling rate. Finally, the decrease in vacancy creation leads to a drop in employment in the next period, and a consequential fall in output.

With the KPR preferences, firms do not necessarily reduce vacancy in order to cut down current wage and thereby counteract the potential profit loss — the drop in current consumption driven by households' precautionary motive already decreases current wage, i.e., w_t^{KPR} is also decreasing in c_t . In fact, under the baseline calibration with the KPR preferences, firms choose to increase vacancy to partly compensate the excessive drop in current wage resulting from the fall of current consumption in response to a positive volatility shock in productivity, leading to a rise of employment in the next period, and eventually an increase in output, see Figure 5.7.

It is still possible, however, for the baseline model with the KPR preferences to generate a decrease in output in response to increased future risk in productivity. One

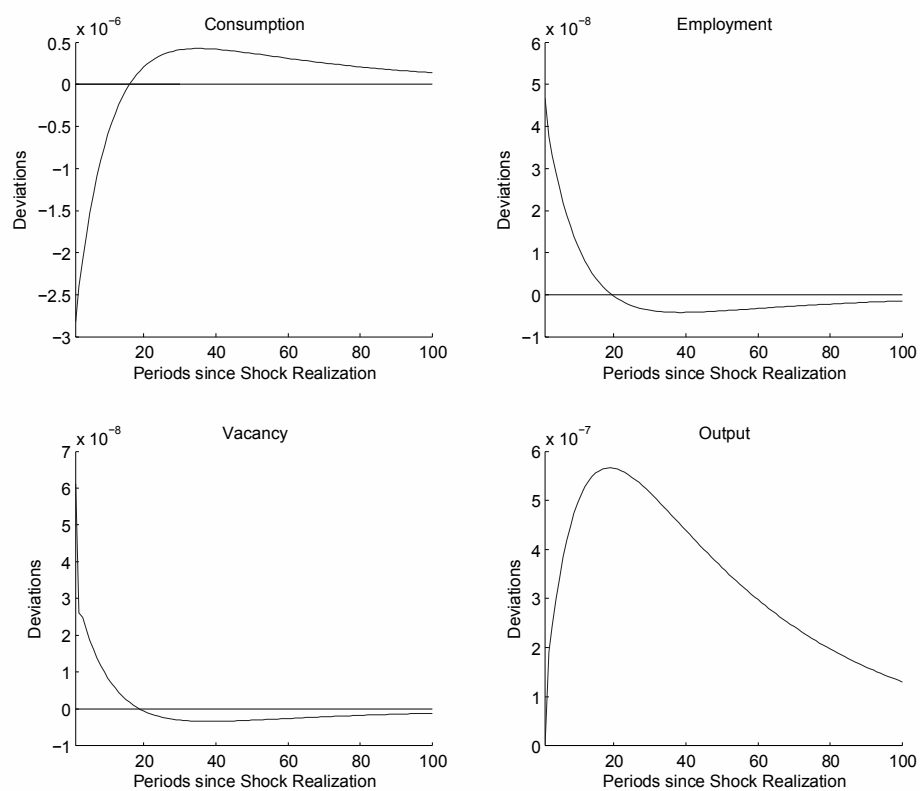


Fig. 5.7 IRF of Macros with KPR Preferences

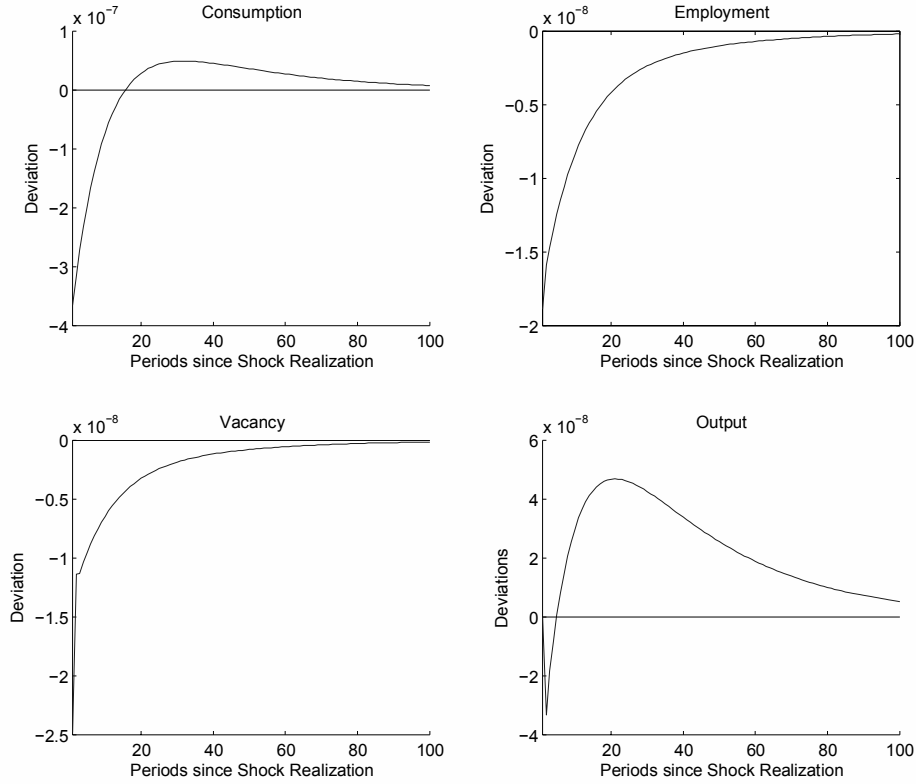


Fig. 5.8 IRF of Macros with KPR Preferences

option is to assume a low level of risk aversion. As is discussed in section 5.4.1, current consumption decreases less with a low κ_F than it would with a high κ_F , and therefore firms still need to cut down current vacancies to ensure a sufficiently large drop in current wage. Then employment in the next period drops and output decreases. See Figure 5.8 for the impulse responses of macro quantities with κ_W setting to one and κ_F to 1 instead of 2 in the baseline calibration.⁹

Note that, when the baseline model is equipped with the GHH preferences, the market wage is determined independently of consumption and therefore becomes less volatile as one source of its volatility has been removed. In other words, the GHH preferences implicitly posit a wage which is less volatile than that associated with the KPR preferences. This observation provides an alternative perspective to understand the propagation mechanism of volatility shock proposed by Basu and Bundick (2012)

⁹ $\kappa_W = 1$ and $\kappa_F = 1$ effectively enforce a special case of the KPR preferences: $U_t = \log c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma}$.

in a monetary model with sticky wage, stick price and the KPR preferences.¹⁰

5.4.2 Moment Comparison

This section examines the contribution from stochastic volatility to the conditional and unconditional volatility of major macroeconomic aggregates respectively. While stochastic volatility can induce a significant amount of additional variations in conditional volatility, its contribution to the unconditional volatility is minimal.

Conditional Variance

The conditional variance of endogenous variables can be expressed as follows

$$\text{var}_t(\mathcal{Y}_{t+1}) = \mathbb{E}_t [(\mathcal{Y}_{t+1} - \mathbb{E}_t \mathcal{Y}_{t+1})(\mathcal{Y}_{t+1} - \mathbb{E}_t \mathcal{Y}_{t+1})'] \quad (5.42)$$

where $\mathbb{E}_t \mathcal{Y}_{t+1}$ denotes the conditional mean. Adding this conditional variance as an additional variable to the vector of endogenous variables and solving the model to third order delivers the third order accurate conditional variance.

Figure 5.9 depicts the simulated time paths of the third order accurate conditional variance of the endogenous variables with and without stochastic volatility (blue and red line respectively). When there is no volatility shock, the conditional variance of all variables exhibit minimal fluctuations along the simulation path. Adding stochastic volatility, however, induces a substantial amount of variations in the conditional variances. This is consistent with the interpretation that volatility shocks are a source of conditional heteroskedasticity, see Andreasen (2012b).

Unconditional Standard Deviation

As noted by Andreasen (2012b), the presence of stochastic volatility may induce additional variation in endogenous variables when a DSGE model is solved to third order. While it is difficult to isolate the effect of volatility shock in a nonlinear environment as noted by Fernández-Villaverde and Rubio-Ramírez (2007), the contribution from volatility shock and from its interaction with level shock to the total unconditional volatility of macroeconomic aggregates can be measured by computing the unconditional standard deviation with and without volatility shock respectively, and then examining the difference.

¹⁰They further send the KPR preferences to the recursive utility framework à la Epstein and Zin, in order to calibrate their model with asset pricing data.

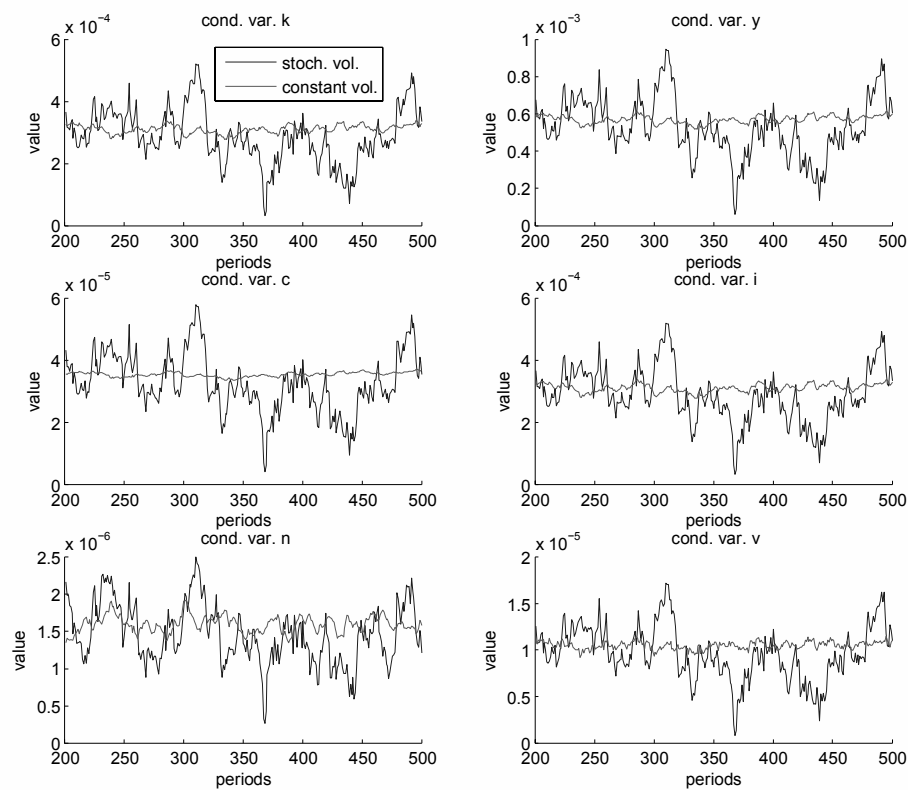


Fig. 5.9 Conditional Variance Comparison, Baseline Model of Section 5.2

Table 5.2 Unconditional standard deviation comparison under Baseline Calibration

variable	constant vol.	stoch. vol.	diff. in %
k	1.1976	1.2202	1.89
y	0.0825	0.0840	1.82
c	0.0456	0.0465	1.97
i	0.0404	0.0411	1.73
n	0.0044	0.0045	2.27
v	0.0037	0.0037	0.00

Table 5.2 documents the unconditional standard deviation of endogenous variables in the absence and presence of volatility shock in productivity (column 2 and 3 respectively), and reports the difference in percentage (last column). Note that the presence of stochastic volatility indeed leads to an increase in the unconditional standard deviation of all endogenous variables, confirming Andreasen's (2012b) simulation-based observation. Such increase, however, is very small across all the variables.

Table 5.3 Unconditional Standard deviation comparison under high risk aversion, higher Frisch elasticity and lower job separation rate

variable	constant vol.	stoch. vol.	diff. in %
k	3.7962	3.8698	1.94
y	0.1430	0.1459	2.03
c	0.0603	0.0616	2.16
i	0.0828	0.0844	1.93
n	0.0075	0.0077	2.67
v	0.0061	0.0062	1.64

Table 5.3 repeats the above unconditional volatility comparison. Yet all the unconditional standard deviations are computed with a higher risk aversion ($\kappa_F = 5$), a higher Frisch elasticity ($1/\gamma = 1.25$) and a lower job separation rate ($\chi = 0.036$), as the preceding discussion has shown that such set of parameter values will enlarge the impact of a volatility shock. Under this risk sensitive calibration, stochastic volatility contributes more to the unconditional volatility of variables than under the baseline calibration (percentage difference in the last column is uniformly larger than that in the last column of Table 5.2). Nevertheless, such contribution is still very small in levels.

5.5 The Extended Model

In this section, I extend the baseline model in section 5.2 to include adjustment cost to investment and variable capital utilization. Jaimovich and Rebelo (2009) show that a general equilibrium model with these two features and the class of preferences with little wealth effect can generate the positive comovement among major macroeconomic aggregates, such as output, consumption, investment and employment in response to a news shock. In the light of their analysis, I show that the extend model also restores the positive comovement particularly between investment and consumption in response to a volatility shock, as argued for in Basu and Bundick (2012).

To facilitate comparison to the results in the literature, I also add consumption habit formation,¹¹ noting that this is not required for the extended model to predict a recession in major macroeconomic aggregates in response to a positive shock in the volatility of productivity. Moreover, I add preferences shock, investment technology shock and government spending shock to the extended model.¹² The volatility of all these three shocks are allowed to change over time.

5.5.1 The Extended Model

With consumption habit formation, variable capital utilization and investment adjustment cost incorporated, the planner faces the following maximization problem¹³

$$\max_{c_t, i_t, v_t, x_t} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left[\frac{e^{b_t} \left(c_t - \kappa_C c_{t-1} - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t \right)^{1-\kappa_F}}{1 - \kappa_F} - 1 \right] \quad (5.43)$$

with

$$S_t = (c_t - \kappa_C c_{t-1})^{\kappa_W} S_{t-1}^{1-\kappa_W} \quad (5.44)$$

where $\kappa_C \in (0, 1)$ governs the persistence of consumption habit and b_t denotes the preferences shock process. The law of motion of capital and production function now take the following form

$$k_{t+1} = (1 - \delta_t) k_t + e^{\mu_t} (1 - \phi_t) i_t \quad (5.45)$$

$$y_t = e^{\tilde{z}_t} (x_t k_t)^{\alpha} n_t^{1-\alpha} \quad (5.46)$$

where δ_t denotes the depreciation function, x_t the capital utilization rate and ϕ_t the investment adjustment cost function. μ_t denotes the investment-specific technology shock process. The depreciation function takes the following functional form as proposed by Baxter and Farr (2005)

$$\delta_t = \frac{\delta_1}{1 + \delta_2} x_t^{1+\delta_2} + \delta_0 \quad (5.47)$$

where δ_0 and δ_1 will be chosen such that capital is fully utilized in the deterministic steady state. δ_2 denotes the elasticity of marginal depreciation with respect to the

¹¹See Bidder and Smith (2012), Christiano et al. (2013) and Born and Pfeifer (2014) for incorporating consumption habit formation in their analysis of volatility shocks in general equilibrium models.

¹²See Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2011a) and Born and Pfeifer (2014) for incorporating these three shocks in analyzing the quantitative impact of volatility shocks in general equilibrium models.

¹³As the model is no longer frictionless, households and firms' problem should be presented and solved separately. Yet for notational ease, I still present the model as a planner's problem, with the same set of equilibrium conditions that would come from the corresponding market model.

utilization rate.

As in Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2011b), Bidder and Smith (2012), Born and Pfeifer (2014) and many others, the investment adjustment cost function takes the following quadratic form

$$\phi_t = \frac{\kappa_I}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \quad (5.48)$$

where κ_I is positive and governs the curvature of the function.

In addition, the government purchases goods and service and balances its budget in each period. This government spending is financed by a lump-sum tax and therefore the resource constraint of the extended model writes

$$y_t = c_t + i_t + \kappa_V v_t + e^{g_t} \quad (5.49)$$

where g_t denotes the government spending and is assumed to be an exogenous process.

Analogous to the productivity process (5.11), the preferences shock process b_t , investment shock process μ_t and the government spending process g_t are driven by their corresponding exogenous innovations with stochastic volatility and take the following form

$$b_t = \rho_b b_{t-1} + e^{\sigma_{b,t}} \varepsilon_{b,t}, \quad \varepsilon_{b,t} \sim \mathcal{N}(0, 1) \quad (5.50)$$

$$\mu_t = \rho_\mu \mu_{t-1} + e^{\sigma_{\mu,t}} \varepsilon_{\mu,t}, \quad \varepsilon_{\mu,t} \sim \mathcal{N}(0, 1) \quad (5.51)$$

$$g_t = (1 - \rho_g) \bar{g} + \rho_g g_{t-1} + e^{\sigma_{g,t}} \varepsilon_{g,t}, \quad \varepsilon_{g,t} \sim \mathcal{N}(0, 1) \quad (5.52)$$

where ρ_b , ρ_μ and ρ_g are persistence parameters, and \bar{g} the deterministic steady state value of government spending. Likewise, analogous to the stochastic volatility process that scales the productivity shock (5.12), the stochastic volatility in the above three processes, $\sigma_{b,t}$, $\sigma_{\mu,t}$ and $\sigma_{g,t}$ are all assumed to take the following form

$$\sigma_{\zeta,t} = (1 - \rho_{\sigma_\zeta}) \bar{\sigma}_\zeta + \rho_{\sigma_\zeta} \sigma_{\zeta,t-1} + \tau_\zeta \omega_{\zeta,t}, \quad \omega_{\zeta,t} \sim \mathcal{N}(0, 1) \quad (5.53)$$

where ρ_{σ_ζ} governs the persistence and $\zeta \in \{b, \mu, g\}$. $\bar{\sigma}_\zeta$ denotes the respective unconditional mean level of $\sigma_{b,t}$, $\sigma_{\mu,t}$ and $\sigma_{g,t}$. τ_ζ scales the volatility shock.

5.5.2 Characterization and Calibration

Defining $\lambda_{1,t}$, $\lambda_{2,t}$, $\lambda_{3,t}$ and $\lambda_{4,t}$ as the Lagrangian multipliers associated with the resource constraint (5.49), the S_t dynamic (5.44), the law of motion of employment (5.6) and capital (5.45) respectively, setting up the associated Lagrangian function and differentiating with respect to the corresponding control and state variables deliver the following set of first order necessary conditions that characterizes the equilibrium of

the extended model

$$\lambda_{1,t} = e^{b_t} \left(c_t - \kappa_C c_{t-1} - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t \right)^{-\kappa_F} + \lambda_{2,t} \kappa_W (c_t - \kappa_C c_{t-1})^{\kappa_W-1} S_{t-1}^{1-\kappa_W} \quad (5.54)$$

$$\begin{aligned} & - \beta \kappa_C \mathbb{E}_t \left[e^{b_{t+1}} \left(c_{t+1} - \kappa_C c_t - \kappa_N \frac{n_{t+1}^{1+\gamma}}{1+\gamma} S_{t+1} \right)^{-\kappa_F} \right] \\ & - \beta \kappa_C \kappa_W \mathbb{E}_t \left[\lambda_{2,t+1} (c_{t+1} - \kappa_C c_t)^{\kappa_W-1} S_t^{1-\kappa_W} \right] \\ \lambda_{2,t} = & - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} \left(c_t - \kappa_C c_{t-1} - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t \right)^{-\kappa_F} \\ & + \beta (1 - \kappa_W) \mathbb{E}_t \left[\lambda_{2,t+1} (c_{t+1} - \kappa_C c_t)^{\kappa_W} S_t^{-\kappa_W} \right] \end{aligned} \quad (5.55)$$

$$\lambda_{3,t} = \lambda_{1,t} \frac{\kappa_V}{m_{v,t}} \quad (5.56)$$

$$\lambda_{1,t} y_{x,t} = \lambda_{4,t} k_t \delta_{x,t} \quad (5.57)$$

$$\begin{aligned} \lambda_{1,t} = & e^{\mu_t} \lambda_{4,t} \left[1 - \kappa_i \left(\frac{i_t}{i_{t-1}} - 1 \right) \frac{i_t}{i_{t-1}} - \frac{\kappa_i}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right] \\ & + \beta \mathbb{E}_t \left[e^{\mu_{t+1}} \lambda_{4,t+1} \kappa_i \left(\frac{i_{t+1}}{i_t} - 1 \right) \left(\frac{i_{t+1}}{i_t} \right)^2 \right] \end{aligned} \quad (5.58)$$

$$\lambda_{4,t} = \beta \mathbb{E}_t [\lambda_{1,t+1} y_{k,t+1} + \lambda_{4,t+1} (1 - \delta_{t+1})] \quad (5.59)$$

$$\lambda_{3,t} = \beta \mathbb{E}_t [U_{n,t+1} + \lambda_{1,t+1} y_{n,t+1} + \lambda_{3,t} (1 - \chi + m_{n,t+1})] \quad (5.60)$$

with $y_{x,t} = \alpha y_t / x_t$, $\delta_{x,t} = \delta_1 x_t^{\delta_2}$, $y_{k,t} = \alpha y_t / k_t$ and $y_{n,t} = (1 - \alpha) y_t / n_t$. $U_{n,t}$, $m_{v,t}$ and $m_{n,t}$ are as defined by (5.27), (5.28) and (5.29). The four remaining first order conditions with respect to the Lagrangian multipliers are the four constraints with which the multipliers are associated.

Among this set of equilibrium conditions, (5.54) and (5.55) define the marginal utility of consumption in the presence of habit formation, and when $\kappa_C = 0$, they reduce to (5.20) and (5.21) respectively. Identical to (5.22), (5.56) denotes the marginal loss in welfare due to vacancy creation in consumption terms. (5.57) characterizes the optimal capital utilization rate by equating the marginal benefit in consumption terms to the marginal cost in terms of additional units of capital being worn out. (5.58) is the Euler equation for investment in the presence of adjustment cost. As in the baseline model, (5.59) and (5.60) are the Euler equations for consumption and employment respectively.

For numerical analysis of the extended model, in addition to the baseline calibration in section 5.3.2, the capital utilization elasticity parameter δ_2 is set to 1, see Basu and

Kimball (1997). Consumption habit persistence κ_C is set to 0.54 as reported in Born and Pfeifer (2014). Given the value of δ_2 and κ_C , the investment adjustment cost elasticity κ_I is accordingly chosen to be 10 such that in response to a positive shock to the volatility of productivity, investment decreases. At the deterministic steady state, government spending \bar{g} is equal to 20% of output as reported in Born and Pfeifer (2014). As a starting point, the persistence and volatility of the preferences shock process, investment shocks process and government spending process are assumed to be the same as those of the productivity process, i.e., $\rho_\zeta = \rho_z = 0.95$, $\rho_{\sigma_\zeta} = \rho_{\sigma_z} = 0.90$, $\overline{\sigma_\zeta} = \overline{\sigma_z} = \ln(0.07)$ and $\tau_\zeta = \tau_z = 0.06$ for $\zeta \in \{b, \mu, g\}$. Owing to the presence of these additional shock processes, the endogenous variables in the extended model are in general more volatile than those in the baseline model. The vacancy posting cost κ_V is thereby set to 0.6, to keep the volatility of labor market tightness relative to that of labor productivity still equal to 7.56. Note that the baseline model is nested in the extended model — when $\kappa_I = \kappa_C = 0$, $\delta_2 \rightarrow \infty$ and all the shocks except the productivity shock shut down, the extended model reduces to the baseline model.

5.5.3 Impulse Responses

This section presents and analyzes the responses of macroeconomic variables to a positive shock in the volatility of productivity, investment technology, preferences and government spending. Except for the volatility of investment technology where a positive shock leads to a boom, an increase in the volatility of all the other three shocks leads to a recession, consistent with the pattern reported by Born and Pfeifer (2014).

Quantitatively, the impact of a volatility shock on the macroeconomic aggregates is very small. For example, under the extended calibration, output deviates from its third order accurate stochastic steady state by about -1.2×10^{-5} in response to a positive, one standard deviation shock in the volatility of productivity. Its responses to such a shock in the volatility of investment technology, preferences and government spending are even smaller in terms of absolute value.

Shock to the Volatility of Productivity

Investment adjustment cost plays an important role in shaping the impulse responses of endogenous variables of the extended model, as summarized by the capital utilization equation (5.57). Inserting the functional form of $y_{x,t}$ and $\delta_{x,t}$ in (5.57) and rearranging

yields

$$1 = \frac{\lambda_{4,t}}{\lambda_{1,t}} x_t^{1-\alpha+\delta_2} \left[\delta_1 k_t^{1-\alpha} (\bar{e}^t n_t)^{\alpha-1} \right] \quad (5.61)$$

where $\lambda_{4,t}/\lambda_{1,t}$ is the value of installed capital in terms of consumption as noted in Jaimovich and Rebelo (2009). Terms inside the bracket are constant and state variables and will not change on impact of volatility shocks. With the presence of adjustment cost to investment, building up a buffer stock of capital in response to a positive volatility shock to productivity by increasing current investment becomes riskier. Instead, manipulating the installed capital on impact is less risky (and possible since utilization rate is a control variable) as the installed capital will not respond to the changes of risk in future productivity, and hence its value in terms of consumption increases on impact. This increase in value makes the installed capital more costly to replace, giving the planner an incentive to slow down the depreciation by lowering the utilization rate and decreasing current investment. Still, driven by the precautionary motive, the planner wants to build up a buffer stock of capital in response to a positive volatility shock to productivity and now it chooses to cut down current consumption to achieve that — the saved stock of current consumption will build up the buffer stock of capital through the resource constraint (5.45) and (5.49) (not pictured) in a less risky manner relative to that through increasing investment as current consumption is not involved in the production process and therefore less sensitive to the change in the volatility of future productivity.

Figure 5.10 depicts the impulse responses of macroeconomic variables, expressed as deviations from their third order accurate stochastic steady states, to a positive, one standard deviation shock in the volatility of productivity, i.e., in $\omega_{z,t}$. As in the baseline model, the decrease in current consumption results in an increase in the marginal utility of consumption. Yet this increase in $\lambda_{1,t}$ is dominated by the increase in the value of installed capital $\lambda_{4,t}$ and therefore $\lambda_{4,t}/\lambda_{1,t}$ increases on impact. The fall in current utilization rate leads to a decrease in effective capital (the lower panel). With productivity having not changed (again, it is only the volatility of the distribution of future productivity shocks that is being shocked) and current employment being fixed, current output in the extended model decline on impact due to this decrease in current effective capital. The increase in the marginal utility of consumption also increases the marginal loss in consumption terms in welfare due to vacancy creation. The planner therefore cuts down current vacancy, leading to a decline in employment in next period. This fall reinforces the decrease in output in the subsequent period and therefore the extended model predicts a deeper and more prolonged recession than the baseline model in response to increased future risk in productivity.

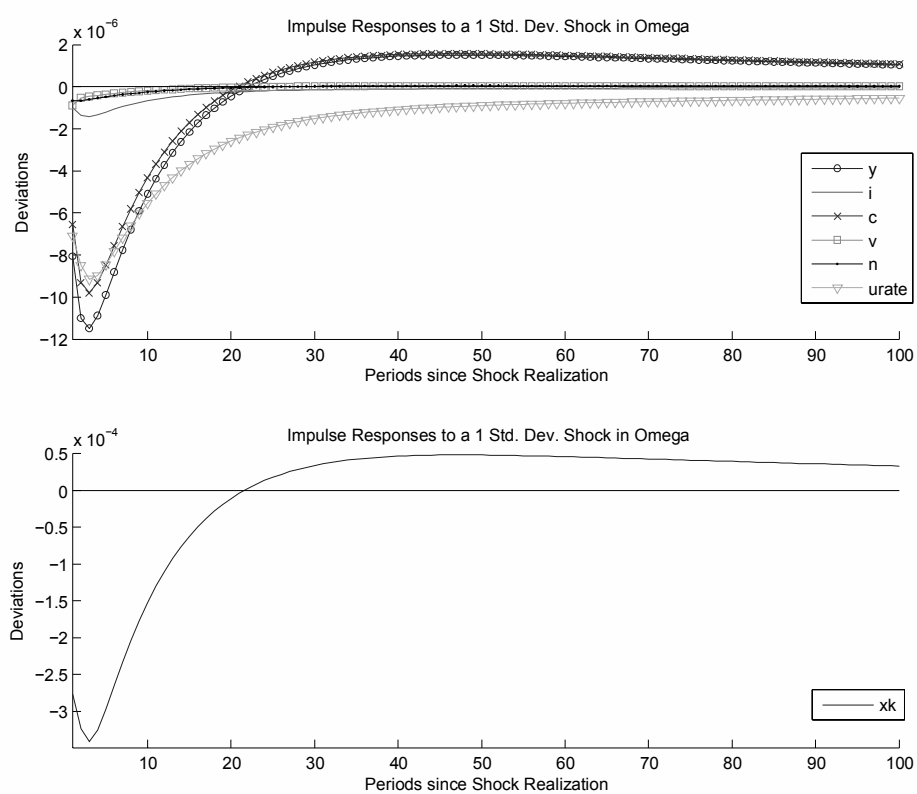


Fig. 5.10 Macro IRFs: Volatility Shock to Productivity, Extended Model of Section 5.5.1

Shock to the Volatility of Investment Technology

To analyze the transmission mechanism of a shock to the volatility of investment technology, it is convenient to interpret the investment level shock $\varepsilon_{\mu,t}$ as the disturbance to the process by which current investment is transformed into installed capital to be used in production, see Justiniano et al. (2010) and Justiniano et al. (2011). When a positive shock hits the volatility of $\varepsilon_{\mu,t}$, the efficiency of this transformation becomes more uncertain, and therefore the planner increases current investment to ensure a sufficient amount of investment will be converted into capital for production purpose. An increase in current investment leads to a fall in the value of installed capital in consumption terms, and as noted by Jaimovich and Rebelo (2009), this fall occurs because adjustment cost to investment implies that higher levels of current investment reduce the cost of investment in the next period. The fall in $\lambda_{4,t}/\lambda_{1,t}$ lowers the value of installed capital, making it less costly to replace, so it is efficient to increase current utilization rate to speed up depreciation.

Figure 5.11 displays the impulse responses of macroeconomic quantities as deviations from their third order accurate stochastic steady state to a positive, one standard deviation shock in the volatility of investment technology, i.e., in $\omega_{i,t}$. As increasing current investment secures a sufficient amount of installed capital for production and of capital input in the next period, the planner chooses to increase current consumption (followed by a decline), leading to a fall in the marginal utility of consumption. This fall in $\lambda_{1,t}$ is dominated by the decline in $\lambda_{4,t}$ and therefore the value of installed capital $\lambda_{4,t}/\lambda_{1,t}$ falls. The increased current utilization rate results in an increase in effective capital (the lower panel), leading to an increase in output on impact. The fall in $\lambda_{1,t}$ also leads to an increase in current vacancy creation and future employment. The latter makes the increase in output even more persistent. In sum, a positive volatility shock to investment technology leads to a boom.

Shock to the Volatility of Preferences and Government Spending

Since both preferences and government spending shocks, i.e. $\varepsilon_{b,t}$ and $\varepsilon_{g,t}$, are demand shocks, a positive shock that hits their volatility leads to a future aggregate demand with high uncertainty. The planner thereby increases its precautionary savings by cutting down current consumption to ensure that future demand can be met.

Figure 5.12 and 5.13 depict the impulse responses of macroeconomic variables, expressed as deviations from their third order accurate stochastic steady states, to a positive, one standard deviation shock in the volatility of preferences and government

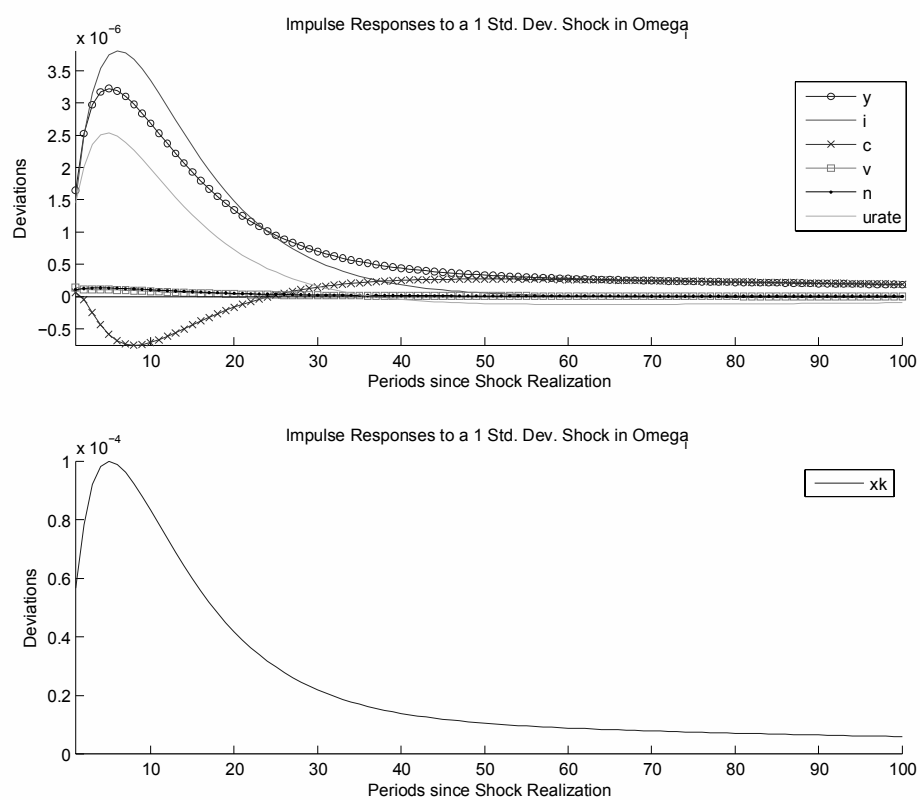


Fig. 5.11 Macro IRFs: Volatility Shock to Investment, Extended Model of Section 5.5.1

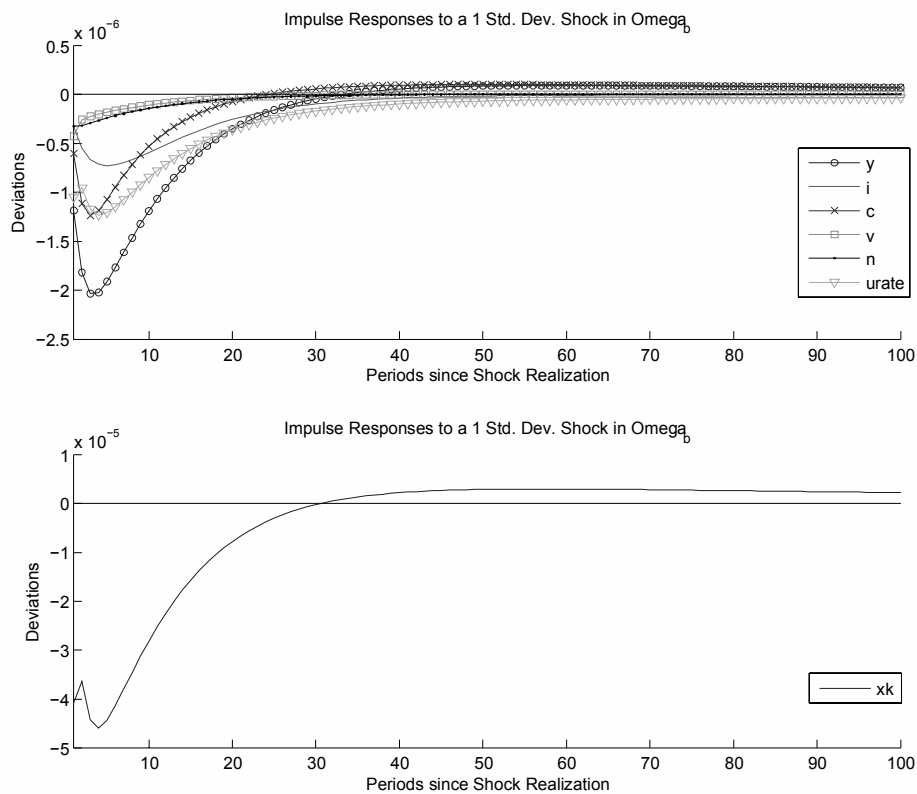


Fig. 5.12 Macro IRFs: Volatility Shock to Preferences, Extended Model of Section 5.5.1

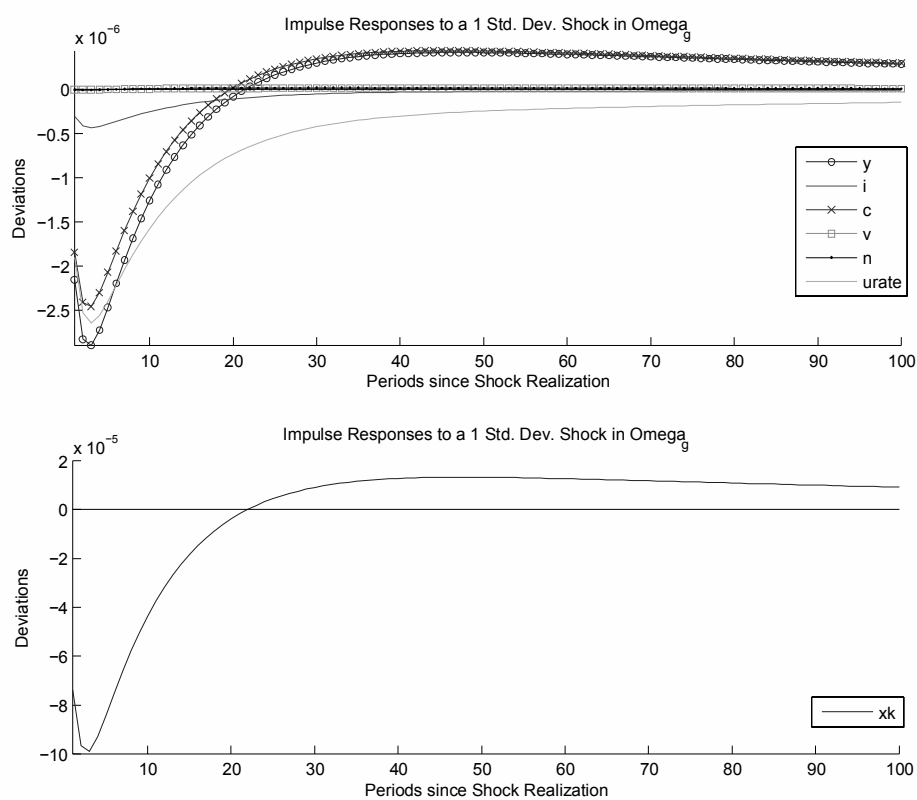


Fig. 5.13 Macro IRFs: Volatility Shock to Government Spending, Extended Model of Section 5.5.1

spending, i.e., in $\omega_{b,t}$ and in $\omega_{g,t}$, respectively. Through a market lens, when future aggregate demand becomes more uncertain, firms choose to rent a smaller amount of effective capital for production purpose from households. As capital is a state variable and being fixed on impact, this decline in the demand of effective capital leads to a fall in current utilization rate. On the other hand, as current consumption has been cut back on impact owing to precautionary motive, a buffer stock of capital will be built using this saved consumption stock in the next period. This crowds out the need of increasing current investment in order to build up the buffer stock of capital. As a result, current investment drops. The fall in current utilization rate leads to a decline in output on impact. The decrease in current consumption leads to an increase in the marginal utility of consumption, a fall in current vacancy and future employment, which reinforces the drop in output in the subsequent period. Put it together, a positive volatility shock to preferences and government spending leads to a recession.

Size of Volatility Shock

In the extended calibration, the standard deviations of the four volatility shocks are all set to be 0.06, and the impulse responses reported in section 5.5.3 - 5.5.3 are generated accordingly. In the literature, the size of these standard deviations vary. For example, the standard deviation of volatility shock in productivity, i.e., τ_z , ranges from 0.01 (see Andreasen (2012b) and Justiniano and Primiceri (2008)), to 0.15 (see Bidder and Smith (2012)), and to 0.312 (see Born and Pfeifer (2014)). Note that, first, a volatility shock of large size will not qualitatively alter the impulse responses of macroeconomic aggregates in the extended model, that is, a large, positive shock in the volatility of productivity still leads to a recession. Second, the responses of macroeconomic aggregates are still small even if the standard deviation of volatility shock is reasonably large. For example, output deviates from its third order accurate stochastic steady state by about -1.2×10^{-4} in response to a positive volatility shock with $\tau_z = 0.624$ in productivity. $\tau_z = 0.624$ mimics the two-standard deviation volatility shock used in Born and Pfeifer (2014) for policy risk study.

5.5.4 Moments

This section presents the unconditional standard deviation of the macroeconomic aggregates in the extended model. Table 5.4 reports the unconditional standard deviations computed in the absence and the presence of stochastic volatility from productivity shock, investment technology shock, preferences shock and government spending

shock (column 2 and 3 respectively). The difference in percentage between these two set of values are shown in the last column.

Table 5.4 Unconditional standard deviation comparison of the extended model

variable	constant vol.	stoch. vol.	diff. in %
k	1.9689	2.0060	1.88
y	0.0800	0.0815	1.88
c	0.0473	0.0482	1.90
i	0.0432	0.0440	1.85
n	0.0032	0.0033	3.12
v	0.0027	0.0027	0.00

Like in the baseline model (see Table 5.2), the contribution from stochastic volatility to the unconditional volatility of the macroeconomic aggregates is very small, although the extended model includes four different sources (instead of one in the baseline model, i.e., the productivity shock) of stochastic volatility. Table 5.5 reports the approximate portion of the total contribution from the four sources of stochastic volatility to the unconditional standard deviation.

The second column of the table repeats the unconditional standard deviations with the presence of all the four sources of stochastic volatility for reference. The third column documents the unconditional standard deviations when the volatility of the productivity shock is hold constant but that of the other three shocks are still allowed to vary over time. Column 4 reports the percentage difference between the previous two columns, and therefore can be considered as the contribution from the stochastic volatility in productivity to the unconditional standard deviation. Analogously, column 5, 7 and 9 documents the unconditional standard deviation without stochastic volatility in investment technology, government spending and preferences respectively, and

Table 5.5 Unconditional standard deviation decomposition of the extended model

variable	stoch. vol.	$\omega_z = 0$		$\omega_i = 0$		$\omega_g = 0$		$\omega_b = 0$	
		Value	Diff.in%	Value	Diff.in%	Value	Diff.in%	Value	Diff.in%
k	2.0060	1.9824	1.19	1.9962	0.49	2.0041	0.09	2.0045	0.07
y	0.0815	0.0803	1.49	0.0814	0.12	0.0815	0.00	0.0815	0.00
c	0.0482	0.0475	1.47	0.0480	0.42	0.0481	0.21	0.0481	0.21
i	0.0440	0.0434	1.38	0.0439	0.23	0.0440	0.00	0.0440	0.00
n	0.0033	0.0033	0.00	0.0033	0.00	0.0033	0.00	0.0033	0.00
v	0.0027	0.0027	0.00	0.0027	0.00	0.0027	0.00	0.0027	0.00

column 6, 8 and 10 reports the contribution in percentage from the three sources of stochastic volatility respectively.

There are two important observations. First, stochastic volatility in productivity and investment technology contributes most to the unconditional standard deviation, yet that in government spending and preferences contributes almost nothing. This is consistent with the observation made in Justiniano and Primiceri (2008) and Fernández-Villaverde et al. (2011b). Second, the percentage contributions shown in column 4, 6, 8 and 10 only approximate the individual contribution from the four sources of stochastic volatility respectively, and thereby not necessarily add up to the percentage contribution reported in the last column of Table 5.4. The remaining contribution comes from the interplay among the volatility shocks and between the level and volatility shocks.

5.6 Conclusion

I have presented a business cycle model that includes Jaimovich and Rebelo's (2009) preferences, search and matching frictions in labor market, investment adjustment cost and variable capital utilization as its key ingredients that are used to explain the propagation mechanism of stochastic volatility in general equilibrium. By construction, the Hosios's (1990) condition holds in the model economy and the frictions induced by search and matching activities in labor market can be internalized. The model thereby encompasses no frictions other than adjustment cost on investment, providing an environment to observe the structural and statistical implications of stochastic volatility almost in isolation.

The model is solved to third order using the nonlinear moving average perturbation, and analyzed under conventional, quarterly calibration. The impulse responses shows that the model predicts a recession in response to a positive shock in the volatility of productivity, government spending and preferences, and envisions a boom if such a positive shock hits the volatility of investment technology, consistent with the pattern reported by Born and Pfeifer (2014) and many other studies in this literature. On the quantitative side, both the impulse responses and unconditional standard deviations suggest that the impact of stochastic volatility on the major macroeconomics aggregates is very small, though stochastic volatility largely increases the conditional volatility of those aggregates. Since the model incorporates no nominal rigidities such as sticky wage and price, the numerical analysis of the model supports the argument that the large impact of stochastic volatility found in Fernández-Villaverde et al. (2011a)

and others using a general equilibrium model with the above rigidities embedded may come from the interaction between stochastic volatility and those nominal frictions.

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Appendix A

Chapter 2 Appendix

A.0.1 Taylor Expansion

The M -th order Taylor approximation of (3.2) at the deterministic steady state (3.8) is

Corollary A.0.1. *An M -th order Taylor Approximation of (3.2) is written as*

$$y_t = \sum_{j=0}^M \frac{1}{j!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^i \sigma^i} \sigma^i \right] (z_t - \bar{z})^{\otimes [j]} \quad (\text{A.1})$$

Proof. From Vetter (1973), a multidimensional Taylor expansion is given by

$$W_{(p \times 1)} \left(\begin{matrix} B \\ (s \times 1) \end{matrix} \right) = W(\bar{B}) + \sum_{n=1}^M \frac{1}{n!} \mathcal{D}_{B^{Tn}}^n W(\bar{B}) (B - \bar{B})^{\otimes [n]} + R_{M+1}(\bar{B}, B) \quad (\text{A.2})$$

$$\text{where } R_{M+1}(\bar{B}, B) = \frac{1}{M!} \int_{\xi=\bar{B}}^B \mathcal{D}_{B^{TM+1}}^{M+1} W(\xi) \left(I_s \otimes (B - \xi)^{\otimes [M]} \right) d\xi \quad (\text{A.3})$$

Differentiating (3.2) M times, a Taylor approximation at the deterministic steady state \bar{z} is

$$\begin{aligned} y_t &= \frac{1}{0!} \left(\frac{1}{0!} \bar{y} + \frac{1}{1!} y_{\sigma} \sigma + \frac{1}{2!} y_{\sigma^2} \sigma^2 + \dots + \frac{1}{M!} y_{\sigma^M} \sigma^M \right) \\ &+ \frac{1}{1!} \left(\frac{1}{0!} y_z + \frac{1}{1!} y_{z\sigma} \sigma + \frac{1}{2!} y_{z\sigma^2} \sigma^2 + \dots + \frac{1}{(M-1)!} y_{z\sigma^{M-1}} \sigma^{M-1} \right) (z_t - \bar{z}) \\ &+ \frac{1}{2!} \left(\frac{1}{0!} y_{z^2} + \frac{1}{1!} y_{z^2\sigma} \sigma + \frac{1}{2!} y_{z^2\sigma^2} \sigma^2 + \dots + \frac{1}{(M-2)!} y_{z^2\sigma^{M-2}} \sigma^{M-2} \right) (z_t - \bar{z})^{\otimes [2]} \\ &\vdots \\ &+ \frac{1}{M!} \frac{1}{0!} y_{z^M} (z_t - \bar{z})^{\otimes [M]} \end{aligned}$$

Writing the foregoing more compactly yields (A.1). □

A.0.2 Projection Appendix

Starting with the capital grid, for any element of the set, $k_t^i \in [k_{min}, k_{max}]$ with i being a positive integer for indexing purpose, the linear transformation

$$\varphi(k_t^i) = \frac{2(k_t^i - k_{min})}{k_{max} - k_{min}} - 1, \quad i = 1, 2, \dots \quad (A.4)$$

ensures that $\varphi(k_t^i)$ is bounded to the set $[-1, 1]$. I choose n_k elements from the set, collected in the vector $k_t = [k_t^1 \ k_t^2 \ \dots \ k_t^{n_k}]'$, such that after applying the linear transformation (A.4) to k_t , the elements of the resulting vector $\varphi(k_t) = [\varphi(k_t^1) \ \varphi(k_t^2) \ \dots \ \varphi(k_t^{n_k})]'$ are the n_k roots of the following n_k th Chebyshev polynomial basis

$$T(\varphi(k_t)) = [T_0 \ T_1(\varphi(k_t)) \ T_2(\varphi(k_t)) \ \dots \ T_{n_k}(\varphi(k_t))] \quad (A.5)$$

where $T_i(\cdot) \equiv \cos(i \arccos(\cdot))$ is the i th Chebyshev polynomial with $T_0 = 1$ and $T(\varphi(k_t))$ is of dimension $(n_k + 1) \times (n_k + 1)$.

Analogous to my choice of elements from the capital set, I choose n_n and n_z elements from these two sets, $n_t = [n_t^1 \ n_t^2 \ \dots \ n_t^{n_n}]'$ and $z = [z_t^1 \ z_t^2 \ \dots \ z_t^{n_z}]'$, that after being transformed by $\varphi(\cdot)$, are the n_n and n_z roots of the following n_n th and n_z th Chebyshev polynomial basis respectively

$$T(\varphi(n_t)) = [T_0 \ T_1(\varphi(n_t)) \ T_2(\varphi(n_t)) \ \dots \ T_{n_n}(\varphi(n_t))] \quad (A.6)$$

$$T(\varphi(z_t)) = [T_0 \ T_1(\varphi(z_t)) \ T_2(\varphi(z_t)) \ \dots \ T_{n_z}(\varphi(z_t))] \quad (A.7)$$

where $T(\varphi(n_t))$ and $T(\varphi(z_t))$ are of dimension $(n_n + 1) \times (n_n + 1)$ and $(n_z + 1) \times (n_z + 1)$ respectively.

As in Judd (1992), Aruoba et al. (2006) and Caldara et al. (2012), the multidimensional basis of the approximated policy function is the Kronecker product of the above three one-dimensional basis

$$X(k_t, n_t, z_t) = T(\varphi(k_t)) \otimes T(\varphi(n_t)) \otimes T(\varphi(z_t)) \quad (A.8)$$

with dimension $(n_g \times n_g)$ where $n_g = (n_k + 1) \times (n_n + 1) \times (n_z + 1)$ is the number of all triplets of the collocation points along three dimensions, i.e., the number of grid points in the three-dimensional state space $[k_{min}, k_{max}] \times [n_{min}, n_{max}] \times [z_{min}, z_{max}]$. With this multidimensional basis, the approximated policy function of consumption and vacancy writes

$$\hat{c}_t = X(k_t, n_t, z_t) \Theta_c = P_c(k_t, n_t, z_t; \Theta_c) \quad (A.9)$$

$$\hat{v}_t = X(k_t, n_t, z_t) \Theta_v = P_v(k_t, n_t, z_t; \Theta_v) \quad (A.10)$$

where $\hat{\cdot}$ indicates these are approximated policy functions, and Θ_c and Θ_v are two vectors of coefficients to be determined. Both \hat{c}_t and \hat{v}_t are of dimension $(n_g \times 1)$.

I solve for the unknown coefficients Θ_c and Θ_v from the two Euler equations (5.23)

and (5.24) using den Haan and Marcet's (1990) functional iteration: at each grid point i

1. use j -th iteration of the coefficients, Θ_c^j and Θ_v^j , to compute

$$n_{t+1}^i = (1 - \chi)n_t^i + m_0 P_v(k_t^i, n_t^i, z_t^i; \Theta_v^j)^{1-\eta} (1 - n_t^i)^\eta, \quad i = 1, 2, \dots, n_g \quad (\text{A.11})$$

$$k_{t+1}^i = (1 - \delta)k_t^i + e^{z_t^i} (k_t^i)^\alpha (n_t^i)^{1-\alpha} - P_c(k_t^i, n_t^i, z_t^i; \Theta_c^j) - \kappa_v P_c(k_t^i, n_t^i, z_t^i; \Theta_v^j) - m_0 \kappa_u (1 - n_t^i) \quad (\text{A.12})$$

$$c_{t+1}^i = P_c(k_{t+1}^i, n_{t+1}^i, \rho z_t^i + \varepsilon_t; \Theta_c^j) \quad (\text{A.13})$$

$$v_{t+1}^i = P_v(k_{t+1}^i, n_{t+1}^i, \rho z_t^i + \varepsilon_t; \Theta_v^j) \quad (\text{A.14})$$

2. given (A.11) - (A.14) and approximating the conditional expectation with the Gauss-Hermite quadrature, the Euler equation for consumption (5.23) writes

$$(\hat{c}_t^i)^{-1} = \beta \sum_{r=1}^m \left[P_c(k_{t+1}^i, n_{t+1}^i, \rho z_t^i + \sqrt{2}\sigma\zeta_r; \Theta_c^j)^{-1} \times \left(1 - \delta + \alpha \exp(\rho z_t^i + \sqrt{2}\sigma\zeta_r) (k_{t+1}^i)^{\alpha-1} (n_{t+1}^i)^{1-\alpha} \right) \frac{\omega_r}{\sqrt{\pi}} \right] \quad (\text{A.15})$$

where ζ_r and ω_r are Gauss-Hermite quadrature points and weights. From the foregoing solve for \hat{c}_t^i . Analogously, the Euler equation for employment (5.24) writes

$$\begin{aligned} (\hat{v}_t^i)^\eta = & \frac{(1 - \eta)m_0}{\kappa_v} (1 - n_t^i)^\eta (\hat{c}_t^i) \beta \\ & \times \sum_{r=1}^m \left[P_c(k_{t+1}^i, n_{t+1}^i, \rho z_t^i + \sqrt{2}\sigma\zeta_r; \Theta_c^j)^{-1} \right. \\ & \times \left(- (n_{t+1}^i)^{-1/\gamma} P_c(k_{t+1}^i, n_{t+1}^i, \rho z_t^i + \sqrt{2}\sigma\zeta_r; \Theta_c^j) \right. \\ & \quad + (1 - \alpha) \exp(\rho z_t^i + \sqrt{2}\sigma\zeta_r) (k_{t+1}^i)^\alpha (n_{t+1}^i)^{-\alpha} + \kappa_u \\ & \quad + \frac{\kappa_v (1 - \chi) P_v(k_{t+1}^i, n_{t+1}^i, \rho z_t^i + \sqrt{2}\sigma\zeta_r; \Theta_v^j)^\eta}{(1 - \eta)m_0 (1 - n_{t+1}^i)^\eta} \\ & \quad \left. \left. - \frac{\eta \kappa_v P_v(k_{t+1}^i, n_{t+1}^i, \rho z_t^i + \sqrt{2}\sigma\zeta_r; \Theta_v^j)}{1 - \eta} \right) \frac{\omega_r}{\sqrt{\pi}} \right] \end{aligned} \quad (\text{A.16})$$

from the foregoing solve for \hat{v}_t^i

3. repeat step 1 - 2 for all n_g grid points, get an estimation of the new coefficients

with the following regression

$$\hat{\Theta}^{j+1} = \begin{bmatrix} \Theta_c^{j+1} & \Theta_v^{j+1} \end{bmatrix} = [X(k_t, n_t, z_t)' X(k_t, n_t, z_t)]^{-1} X(k_t, n_t, z_t)' \begin{bmatrix} \hat{c}_t & \hat{v}_t \end{bmatrix} \quad (\text{A.17})$$

where $X(k_t, n_t, z_t)$ is the multidimensional basis defined by (A.8). Then obtain the $(j+1)$ -th iteration of the coefficients with the following updating rule

$$\Theta^{j+1} = \Lambda \hat{\Theta}^{j+1} + (1 - \Lambda) \Theta^j \quad (\text{A.18})$$

where $\Lambda \in (0, 1]$ is a parameter for stabilizing the iteration.

4. repeat step 1-3 till $\|\Theta^{j+1} - \Theta^j\|$ is smaller than a desired level of tolerance.

The choice of parameters for the iteration is summarized in Table A.1.

Table A.1 Parameters of the Iteration

	symbol	value	source
Number of collocation points for capital	n_k	11	Aruoba et al. (2006)
Number of collocation points for employment	n_n	11	
Number of collocation points for productivity	n_z	9	Aruoba et al. (2006)
Number of Gauss-Hermite points	m	9	Judd (1992)
Tolerance for convergence		$1e - 14$	

Appendix B

Chapter 3 Appendix

B.0.3 Matrix Calculus and Taylor Expansion

Matrix Calculus Definition

Definition B.0.2. *Matrix Derivative and Commutation Matrix*

1. *Matrix Derivative [See Vetter (1973).]*

$$\mathcal{D}_{b_{kl}} A(B) \equiv \left[\frac{\partial a_{ij}}{\partial b_{kl}} \right]_{p \times q} = \begin{bmatrix} \frac{\partial a_{11}}{\partial b_{kl}} & \dots & \frac{\partial a_{1q}}{\partial b_{kl}} \\ \vdots & & \vdots \\ \frac{\partial a_{p1}}{\partial b_{kl}} & \dots & \frac{\partial a_{pq}}{\partial b_{kl}} \end{bmatrix} \quad (\text{B.1})$$

$$\mathcal{D}_B A(B) \equiv [\mathcal{D}_{b_{kl}} A(B)]_{sp \times tq} = \begin{bmatrix} \mathcal{D}_{b_{11}} A(B) & \dots & \mathcal{D}_{b_{1t}} A(B) \\ \vdots & & \vdots \\ \mathcal{D}_{b_{s1}} A(B) & \dots & \mathcal{D}_{b_{st}} A(B) \end{bmatrix} \quad (\text{B.2})$$

2. *Commutation Matrix $K_{a,b}$ [See Magnus and Neudecker's (1979) Theorem 3.1.]*

$$B \otimes A = K_{m,n} (A \otimes B) K_{s,t} \quad (\text{B.3})$$

$m \times t \quad n \times s$

Proof of theorem 3.2.3

1. **Matrix Product Rule:** Combine Vetter's (1973) transpose and product rules and examine the special case of an underlying vector variable.
2. **Matrix Chain Rule:** Combine Vetter's (1973) transpose and chain rules and examine the special case of an underlying vector variable.
3. **Matrix Kronecker Product Rule:** Combine Vetter's (1973) transpose and Kro-

necker rules¹ with an underlying vector variable. Commute the term $H_b \otimes F$ and note that

$$K_{q,vs} \left(\begin{matrix} I \\ s \times s \end{matrix} \otimes K_{v,q} \right) = \left(K_{q,s} \otimes \begin{matrix} I \\ v \times v \end{matrix} \right) \left(\begin{matrix} I \\ s \times s \end{matrix} \otimes K_{q,v} \right) \left(\begin{matrix} I \\ s \times s \end{matrix} \otimes K_{v,q} \right) = \left(K_{q,s} \otimes \begin{matrix} I \\ v \times v \end{matrix} \right) \quad (\text{B.4})$$

where the first equality follows from Henderson and Searle (1981, p. 285) and the second from Magnus and Neudecker (1979, p. 383).

Proof of corollary A.0.1

From Vetter (1973, pp. 358–363), a multidimensional Taylor expansion using the structure of derivatives (evaluated at \bar{B}) in appendix B.0.3 is given by

$$\begin{matrix} M \\ (p \times 1) \end{matrix} \left(\begin{matrix} B \\ (s \times 1) \end{matrix} \right) = M(\bar{B}) + \sum_{n=1}^N \frac{1}{n!} \mathcal{D}_{B^{T^n}} M(\bar{B}) (B - \bar{B})^{\otimes [n]} + R_{N+1}(\bar{B}, B) \quad (\text{B.5})$$

$$\text{where } R_{N+1}(\bar{B}, B) = \frac{1}{N!} \int_{\xi=\bar{B}}^B \mathcal{D}_{B^{T^{N+1}}} M(\xi) \left(I_s \otimes (B - \xi)^{\otimes [N]} \right) d\xi \quad (\text{B.6})$$

Differentiating (3.2) with respect to all its arguments M times, evaluating at the steady state \bar{y} , and noting permutations of the order of differentiation, a Taylor approximation is

$$\begin{aligned} y_t &= \frac{1}{0!} \left(\frac{1}{0!} \bar{y} + \frac{1}{1!} y_{\sigma} \sigma + \frac{1}{2!} y_{\sigma^2} \sigma^2 + \dots + \frac{1}{M!} y_{\sigma^M} \sigma^M \right) \\ &+ \frac{1}{1!} \sum_{i_1=0} \left(\frac{1}{0!} y_{i_1} + \frac{1}{1!} y_{\sigma i_1} \sigma + \frac{1}{2!} y_{\sigma^2 i_1} \sigma^2 + \dots + \frac{1}{(M-1)!} y_{\sigma^{M-1} i_1} \sigma^{M-1} \right) \varepsilon_{t-i_1} \\ &+ \frac{1}{2!} \sum_{i_1=0} \sum_{i_2=0} \left(\frac{1}{0!} y_{i_1 i_2} + \frac{1}{1!} y_{\sigma i_1 i_2} \sigma + \frac{1}{2!} y_{\sigma^2 i_1 i_2} \sigma^2 + \dots + \frac{1}{(M-2)!} y_{\sigma^{M-2} i_1 i_2} \sigma^{M-2} \right) \varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \\ &\vdots \\ &+ \frac{1}{M!} \sum_{i_1=0} \sum_{i_2=0} \dots \sum_{i_m=0} \frac{1}{0!} y_{i_1 i_2 \dots i_m} \varepsilon_{t-i_1} \otimes \varepsilon_{t-i_2} \otimes \dots \otimes \varepsilon_{t-i_m} \end{aligned}$$

Writing the foregoing more compactly yields (A.1) in the text.

¹Note Brewer's (1978) correction to Vetter's (1973) Kronecker rule.

B.0.4 Auxiliary Matrices

Shifting Matrices

$$\delta_1 = \begin{bmatrix} \alpha & \beta_1 \\ ny \times ny & ny \times ne \\ 0 & N \\ ne \times ny & ne \times ne \end{bmatrix} \quad \delta_2 = \begin{bmatrix} \alpha & \beta_2 \\ 0 & \delta_1 \otimes \delta_1 \end{bmatrix} \quad \delta_3 = \begin{bmatrix} \delta_1 \otimes \delta_1 \otimes \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 \otimes \delta_1 & 0 & 0 \\ 0 & 0 & \delta_1 \otimes \delta_2 & 0 \\ 0 & 0 & 0 & \delta_1 \otimes \delta_2 \end{bmatrix} \quad (\text{B.7})$$

$$\gamma_1 = \begin{bmatrix} I & 0 \\ ny \times ny & ny \times ne \\ \alpha & \beta_1 \\ \alpha^2 & \alpha\beta_1 + \beta_1 N \\ 0 & I \\ ne \times ny & ne \times ne \end{bmatrix} \quad \gamma_2 = \begin{bmatrix} I & 0 \\ ny \times ny & ny \times (ny+ne)^2 \\ \alpha & \beta_2 \\ \alpha^2 & \alpha\beta_2 + \beta_2(\delta_1 \otimes \delta_1) \\ 0 & 0 \\ ne \times ny & ne \times (ny+ne)^2 \end{bmatrix} \quad (\text{B.8})$$

$$\gamma_3 = \begin{bmatrix} \gamma_1 \otimes \gamma_1 \otimes \gamma_1 & 0 & 0 & 0 \\ 0 & \gamma_2 \otimes \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_1 \otimes \gamma_2 & 0 \\ 0 & 0 & 0 & \gamma_1 \otimes \gamma_2 \end{bmatrix} \quad \gamma_4 = \begin{bmatrix} 0 \\ ny \times ny \\ 0 \\ ny \times ny \\ I \\ ny \times ny \\ 0 \\ ne \times ny \end{bmatrix} \quad (\text{B.9})$$

$$\gamma_5 = \begin{bmatrix} I_{(ny+ne)^3} \\ \begin{bmatrix} 0 \\ ny \times (ny+ne)^2 \\ I_{(ny+ne)^2} \end{bmatrix} \otimes I_{(ny+ne)} \\ I_{(ny+ne)} \otimes \begin{bmatrix} 0 \\ ny \times (ny+ne)^2 \\ I_{(ny+ne)^2} \end{bmatrix} (K_{(ny+ne), (ny+ne)} \otimes I_{(ny+ne)}) \\ I_{(ny+ne)} \otimes \begin{bmatrix} 0 \\ ny \times (ny+ne)^2 \\ I_{(ny+ne)^2} \end{bmatrix} \end{bmatrix} \quad (\text{B.10})$$

State Spaces for the Markov Representation

$$x_i = \gamma_1 S_i, S_i = \begin{bmatrix} y_{i-1} \\ u_i \end{bmatrix}, \text{ and } S_{i+1} = \delta_1 S_i \quad (\text{B.11})$$

$$x_{j,i} = \gamma_2 S_{j,i}, S_{j,i} = \begin{bmatrix} y_{j-1,i-1} \\ S_j \otimes S_i \end{bmatrix}, \text{ and } S_{j+1,i+1} = \delta_2 S_{j,i} \quad (\text{B.12})$$

$$S_{k,j,i} = \begin{bmatrix} S_k \otimes S_j \otimes S_i \\ S_{k,j} \otimes S_i \\ (S_j \otimes S_{k,i})(K_{ne,ne} \otimes I_{ne}) \\ S_k \otimes S_{j,i} \end{bmatrix} \text{ and } S_{k+1,j+1,i+1} = \delta_3 S_{k,j,i} \quad (\text{B.13})$$

B.0.5 Details of Third-Order Derivation

We begin by differentiating f with respect to each triplet of shocks. The resulting system of equations remains linear in the third derivatives

$$\begin{aligned} \mathcal{D}_{\varepsilon_{t-k}^T \varepsilon_{t-j}^T \varepsilon_{t-i}^T} f &= f_{x^3}(x_k \otimes x_j \otimes x_i) + f_{x^2}(x_{k,j} \otimes x_i) \\ &\quad + f_{x^2}(x_j \otimes x_{k,i})(K_{ne,ne} \otimes I_{ne}) + f_{x^2}(x_k \otimes x_{j,i}) + f_x x_{k,j,i} \end{aligned} \quad (\text{B.14})$$

Evaluating this at \bar{x} and setting its expectation to zero yields

$$\begin{aligned} E_t(\mathcal{D}_{\varepsilon_{t-k}^T \varepsilon_{t-j}^T \varepsilon_{t-i}^T} f) \Big|_{\bar{x}} &= f_{y^-} y_{k-1,j-1,i-1} + f_y y_{k,j,i} + f_{y^+} y_{k+1,j+1,i+1} \\ &\quad + f_{x^3}(x_k \otimes x_j \otimes x_i) + f_{x^2}(x_{k,j} \otimes x_i) \\ &\quad + f_{x^2}(x_j \otimes x_{k,i})(K_{ne,ne} \otimes I_{ne}) + f_{x^2}(x_k \otimes x_{j,i}) \\ &= 0, \text{ for } k, j, i = 0, 1, \dots, \text{ with } y_{k,j,i} = 0, \text{ for } k, j, i < 0 \end{aligned} \quad (\text{B.15})$$

a linear deterministic second order difference equation in the third derivative $y_{k,j,i}$. The homogeneous components in (B.15) are identical to those in (3.18) and (3.26). The inhomogeneous components again have a first order Markov representation. Using the shifting and transition matrices defined in appendix B.0.4 gives (3.40) of the main text, whose solution takes the form (3.41). By recursively substituting (3.41) in (B.15), using the shifting matrices and matching coefficients, we obtain a Sylvester equation in β_3

$$(f_y + f_{y^+} \alpha) \beta_3 + f_{y^+} \beta_3 \delta_3 = - \begin{bmatrix} f_{x^3} & f_{x^2} & f_{x^2} & f_{x^2} \end{bmatrix} \gamma_3 \quad (\text{B.16})$$

Now we move on to the partial derivatives of y function involving the perturbation

parameter σ . To determine $y_{\sigma,j,i}$, we differentiate f with respect to ε_{t-i} , ε_{t-j} and σ

$$\begin{aligned} \mathcal{D}_{\sigma \varepsilon_{t-j}^T \varepsilon_{t-i}^T} f &= f_{x^3}(\mathcal{D}_{\sigma} x \otimes x_j \otimes x_i) + f_{x^2}(\mathcal{D}_{\sigma} x \otimes x_{j,i}) + f_{x^2}(\mathcal{D}_{\sigma} x_j \otimes x_i) \\ &\quad + f_{x^2}(x_j \otimes \mathcal{D}_{\sigma} x_i) K_{ne,ne} + f_x \mathcal{D}_{\sigma} x_{j,i} \\ &\quad \text{where } \mathcal{D}_{\sigma} x_{j,i} = x_{\sigma,j,i} + x_{\tilde{\varepsilon},j,i}(\varepsilon_{t+1} \otimes I_{ne^2}) \end{aligned} \quad (\text{B.17})$$

Evaluating at \bar{x} , taking expectations, setting the resulting expression to zero yields, and noting the results from lower orders yields the expression in the text, whose solution, again analogously to lower orders, takes the form $y_{\sigma,j,i} = \alpha y_{\sigma,j-1,i-1}$, with $y_{\sigma,j,i} = 0$, for $j, i < 0$ delivering (3.43) in the main text.

To determine $y_{\sigma^2,i}$, we differentiate f with respect to ε_{t-i} once and σ twice

$$\begin{aligned} \mathcal{D}_{\sigma^2 \varepsilon_{t-i}^T} f &= f_{x^3}(\mathcal{D}_{\sigma} x \otimes \mathcal{D}_{\sigma} x \otimes x_i) + f_{x^2}(\mathcal{D}_{\sigma} x \otimes \mathcal{D}_{\sigma} x_i) + f_{x^2}(\mathcal{D}_{\sigma^2} x \otimes x_i) + f_{x^2}(\mathcal{D}_{\sigma} x \otimes \mathcal{D}_{\sigma} x_i) + f_x \mathcal{D}_{\sigma^2} x_i \\ &\quad \text{where } \mathcal{D}_{\sigma^2} x_i = x_{\sigma^2,i} + 2x_{\sigma,\tilde{\varepsilon},i}(\varepsilon_{t+1} \otimes I_{ne}) + x_{\tilde{\varepsilon}^2,i}(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes I_{ne}) \end{aligned} \quad (\text{B.18})$$

Evaluating at the deterministic steady state (\bar{x}), taking expectations, and setting the resulting expression to zero yields the expression in the main text, which is still a second order deterministic difference equation. The homogeneous components are packed in $x_{\sigma^2,i}$, and they are identical to those in (3.18) and (3.26). The inhomogeneous components can again be rearranged to have a first order Markov representation by using the shifting and transition matrices defined in appendix B.0.4, thus

$$\begin{aligned} &f_{y^-} y_{\sigma^2,i-1} + f_y y_{\sigma^2,i} + f_{y^+} y_{\sigma^2,i+1} \\ &+ \left\{ \left[f_{x^3}(\gamma_4 \beta_1 \otimes \gamma_4 \beta_1 \otimes \gamma_1) + f_{x^2}([\gamma_4 \beta_2(S_0 \otimes S_0)] \otimes \gamma_1) + 2f_{x^2}(\gamma_4 \beta_1 \otimes [\gamma_4 \beta_2(S_0 \otimes \delta_1)]) \right. \right. \\ &\quad \left. \left. + f_x \gamma_4 \beta_3 \gamma_5(S_0 \otimes S_0 \otimes \delta_1) \right] [E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) \otimes I_{ny+ne}] + f_{x^2}(x_{\sigma^2} \otimes \gamma_1) \right\} S_i = 0 \end{aligned} \quad (\text{B.19})$$

for $i = 0, 1, \dots$, with $y_{-1} = 0$

The solution of the foregoing takes the form of (3.45) in the main text Substituting (3.45) in (B.19) and matching coefficients, we obtain a Sylvester equation in β_{σ}

$$\begin{aligned} (f_y + f_{y^+} \alpha) \beta_{\sigma} + f_{y^+} \beta_{\sigma} \delta_1 &= - \left\{ \left[f_{x^3}(\gamma_4 \beta_1 \otimes \gamma_4 \beta_1 \otimes \gamma_1) + f_{x^2}([\gamma_4 \beta_2(S_0 \otimes S_0)] \otimes \gamma_1) \right. \right. \\ &\quad \left. \left. + 2f_{x^2}(\gamma_4 \beta_1 \otimes [\gamma_4 \beta_2(S_0 \otimes \delta_1)]) + f_x \gamma_4 \beta_3 \gamma_5(S_0 \otimes S_0 \otimes \delta_1) \right] [E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) \otimes I_{ny+ne}] + f_{x^2}(x_{\sigma^2} \otimes \gamma_1) \right\} \end{aligned} \quad (\text{B.20})$$

To determine y_{σ^3} , we differentiate f with respect to σ three times

$$\mathcal{D}_{\sigma^3}f = f_{x^3}(\mathcal{D}_{\sigma}x \otimes \mathcal{D}_{\sigma}x \otimes \mathcal{D}_{\sigma}x) + 2f_{x^2}(\mathcal{D}_{\sigma}x \otimes \mathcal{D}_{\sigma^2}x) + f_{x^2}(\mathcal{D}_{\sigma^2}x \otimes \mathcal{D}_{\sigma}x) + f_x \mathcal{D}_{\sigma^3}x \quad (\text{B.21})$$

where $\mathcal{D}_{\sigma^3}x = x_{\sigma^3} + 3x_{\sigma^2, \tilde{\varepsilon}}\varepsilon_{t+1} + 3x_{\sigma, \tilde{\varepsilon}^2}(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) + x_{\tilde{\varepsilon}^3}(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})$

Evaluating this at the deterministic steady state and setting its expectation to zero yields

$$\begin{aligned} E_t(\mathcal{D}_{\sigma^3}f) \Big|_{\bar{x}} &= f_{x^3}[(x_{\tilde{\varepsilon}} \otimes x_{\tilde{\varepsilon}} \otimes x_{\tilde{\varepsilon}})E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})] + 2f_{x^2}[(x_{\tilde{\varepsilon}} \otimes x_{\tilde{\varepsilon}^2})E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})] \\ &\quad + f_{x^2}[(x_{\tilde{\varepsilon}^2} \otimes x_{\tilde{\varepsilon}})E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})] + f_x[y_{\sigma^3} + x_{\tilde{\varepsilon}^3}E_t(\varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1})] \\ &= 0 \end{aligned} \quad (\text{B.22})$$

using Magnus and Neudecker's (1979) commutation matrix to combine the terms with f_{x^2} as the leading coefficient yields the expression in the text.

B.1 Appendices

B.1.1 Proof of Lemma 4.2.1

As (4.1) is linear in ω_t , only the zeroth and first derivatives with respect to ω_t can be nonzero. As we approximate around the point $\omega_t = 0$, all derivatives of zeroth order with respect to ω_t are zero. Hence, the Taylor expansion of (4.1) is the product of

$\omega_t \frac{\partial e^{\zeta_t} \omega_t}{\partial \omega_t} \bigg|_{\omega_t = \zeta_t = 0}$ and the Taylor series of e^{ζ_t} around the point $\zeta_t = 0$, which is given by $\sum_{i=0}^{\infty} \frac{1}{i!} \zeta_t^i$.

B.1.2 Proof of Proposition 4.2.2

z_t^ω enters into the model (4.3) through the term $He^{\zeta_t} \omega_t$, which itself enters the model linearly. Hence and following lemma 4.2.1, terms through g_{z^ω} , g_{zz^ω} , and $g_{z^2z^\omega}$ are independent of stochastic volatility. Additionally, that z_t^ω enters into the model only through the term $He^{\zeta_t} \omega_t$, which enters linearly, means that terms of the form $g_{z^\omega 2}$ are given by $(g_{\zeta\omega} \zeta_{t-1} + g_{\eta\omega} \eta_t) \omega_t$ following lemma 4.2.1 and the first order autoregressive definition of the volatility process ζ_t in (4.4). These terms dependent on stochastic volatility interact with all the states, z_t , at the next order—accordingly terms of the form $g_{z,z^\omega 2}$ are given by $\frac{1}{2} (g_{\zeta\omega z} \zeta_{t-1} + g_{\eta\omega z} \eta_t) \omega_t z_t$ —and with the states of the volatility process, ζ_{t-1} and η_t ,²—hence, terms of the form $g_{z^\omega 3}$ are given by $\frac{1}{2} (g_{\zeta^2\omega} \zeta_{t-1}^2 + 2g_{\zeta\eta\omega} \zeta_{t-1} \eta_t + g_{\eta^2\omega} \eta_t^2) \omega_t$.

Finally, turning to terms involving the perturbation parameter, σ , terms through $g_{\sigma z^j z^\omega, k}$, i.e., first order in σ , are zero following Jin and Judd (2002), Schmitt-Grohé and Uribe (2004), and others. Likewise, g_{σ^3} will be zero following our assumption of normality of all exogenous processes, see Andreasen (2012b) for an investigation of the consequences of nonnormality. This leaves terms through, g_{σ^2} , $g_{\sigma^2 z}$, and $g_{\sigma^2 z^\omega}$ as claimed in the proposition. That these terms are in general dependent on stochastic volatility will be addressed in the proof of corollary 4.2.5 as there we will examine a condition under which these terms can explicitly be shown to be independent of stochastic volatility.

²But not the shock ω_t , as it enters the model (4.3) linearly through the term $He^{\zeta_t} \omega_t$ and, as such, only terms first order in it are nonzero, see Lemma 4.2.1.

B.1.3 Proof of Corollary 4.2.5

First, the derivatives with respect to ζ_{t-1} . Recall from (4.4) that $\zeta_t = \rho \zeta_{t-1} + \tau \eta_t$.

$$\begin{aligned} 0 &= \mathcal{D}_{\zeta_{t-1}} \{f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + H e^{\zeta_t} \omega_t\} \\ &= f_{y^+} y_y^+ y_\zeta + f_{y^+} y_\zeta^+ \rho + f_y y_\zeta + H e^{\zeta_t} \omega_t \rho \end{aligned} \quad (\text{B.23})$$

at the deterministic steady state, $y^+ = y$ and $\omega_t = 0$ which gives

$$0 = (f_{y^+} y_y + f_{y^+} y_\zeta \rho + f_y) y_\zeta \Rightarrow y_\zeta = 0 \quad (\text{B.24})$$

where the invertibility of $(f_{y^+} y_y + f_{y^+} y_\zeta \rho + f_y)$ follows from the stability of y_y and ρ , see Lan and Meyer-Gohde (2013c).

Now differentiate (B.23) twice with respect to σ , the perturbation parameter,

$$\begin{aligned} 0 &= \mathcal{D}_{\sigma^2 \zeta_{t-1}} \{f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + H e^{\zeta_t} \omega_t\} \\ &= \mathcal{D}_{\sigma^2} \left\{ (f_{y^+} y_y^+ + f_y) y_\zeta + f_{y^+} y_\zeta^+ \rho + H e^{\zeta_t} \omega_t \rho \right\} \\ &= \mathcal{D}_{\sigma^2} \left\{ (f_{y^+} y_y^+ + f_y) \right\} y_\zeta + 2 \mathcal{D}_{\sigma} \left\{ (f_{y^+} y_y^+ + f_y) \right\} y_{\sigma \zeta} + (f_{y^+} y_y^+ + f_y) y_{\sigma^2 \zeta} \\ &\quad + \mathcal{D}_{\sigma^2} \left\{ f_{y^+} \right\} y_\zeta^+ \rho + 2 \mathcal{D}_{\sigma} \left\{ f_{y^+} \right\} \mathcal{D}_{\sigma} \left\{ y_\zeta^+ \right\} \rho + f_{y^+} \mathcal{D}_{\sigma^2} \left\{ y_\zeta^+ \right\} \rho \end{aligned} \quad (\text{B.25})$$

From (B.24), $y_\zeta = y_\zeta^+ = 0$ in a steady state; following Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004), first order derivatives with respect to σ are likewise zero— $y_{\sigma \zeta} = 0$. Thus, if $\rho = 0$, the foregoing is³

$$(f_{y^+} y_y^+ + f_y) y_{\sigma^2 \zeta} = 0 \Rightarrow y_{\sigma^2 \zeta} = 0 \quad (\text{B.27})$$

Turning finally to the derivatives with respect to η_t . Recall again from (4.4) that $\zeta_t = \rho \zeta_{t-1} + \tau \eta_t$

$$\begin{aligned} 0 &= \mathcal{D}_{\eta_t} \{f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + H e^{\zeta_t} \omega_t\} \\ &= f_{y^+} y_y^+ y_\eta + f_{y^+} y_\zeta^+ \tau + f_y y_\eta + H e^{\zeta_t} \omega_t \tau \end{aligned} \quad (\text{B.28})$$

at the deterministic steady state, $y^+ = y$, $\omega_t = 0$, and $y_\zeta = 0$ which gives

$$0 = (f_{y^+} y_y + f_y) y_\eta \Rightarrow y_\eta = 0 \quad (\text{B.29})$$

where the invertibility of $(f_{y^+} y_y + f_y)$ follows from the stability of y_y , see Lan and Meyer-Gohde (2013c).

³This begs the question, whether $y_{\sigma^2 \zeta}$ can ever be different from zero; i.e. whether stochastic volatility can ever have an effect through the risk channel. If $\rho \neq 0$, the conditional expectation of (B.25) at the deterministic steady state is

$$y_{\sigma^2 \zeta} = - (f_{y^+} y_y + f_{y^+} \rho + f_y)^{-1} \left(f_{y^+} y_{\varepsilon \omega^2 \zeta} E_t \left[\varepsilon_{t+1}^{\omega \otimes [2]} \right] + 2 f_{y^+ 2} (y_{\omega} \otimes y_{\omega \zeta}) E_t \left[\omega_{t+1}^{\otimes [2]} \right] \right) \rho \quad (\text{B.26})$$

Now differentiate (B.28) twice with respect to σ , the perturbation parameter,

$$\begin{aligned}
0 &= \mathcal{D}_{\sigma^2 \eta_t} \{f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) + H e^{\zeta_t} \omega_t\} \\
&= \mathcal{D}_{\sigma^2} \left\{ (f_{y^+} y_y^+ + f_y) y_\eta + f_{y^+} y_\zeta^+ \tau + H e^{\zeta_t} \omega_t \tau \right\} \\
&= \mathcal{D}_{\sigma^2} \left\{ (f_{y^+} y_y^+ + f_y) \right\} y_\eta + 2 \mathcal{D}_\sigma \left\{ (f_{y^+} y_y^+ + f_y) \right\} y_{\sigma \eta} + (f_{y^+} y_y^+ + f_y) y_{\sigma^2 \eta} \\
&\quad + \mathcal{D}_{\sigma^2} \left\{ f_{y^+} \right\} y_\zeta^+ \tau + 2 \mathcal{D}_\sigma \left\{ f_{y^+} \right\} \mathcal{D}_\sigma \left\{ y_\zeta^+ \right\} \tau + f_{y^+} \mathcal{D}_{\sigma^2} \left\{ y_\zeta^+ \right\} \tau \quad (B.30)
\end{aligned}$$

From (B.29), $y_\eta = 0$ in a steady state, likewise $y_\zeta = y_\zeta^+ = 0$ from (B.24); following Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004), first order derivatives with respect to σ are likewise zero— $y_{\sigma \zeta} = 0$ and $y_{\sigma \eta} = 0$. If $\rho = 0$, σ_{t-1} vanishes from the systems of equations and all derivatives with respect to it are equal to zero: $\mathcal{D}_{\sigma^2} \left\{ y_\zeta^+ \right\} = \mathcal{D}_\sigma \left\{ y_\zeta^+ \right\} = 0$; accordingly the foregoing is⁴

$$(f_{y^+} y_y^+ + f_y) y_{\sigma^2 \eta} = 0 \Rightarrow y_{\sigma^2 \eta} = 0 \quad (B.32)$$

B.1.4 Detrending the Model

Stationary consumption, investment, capital stock and output, denoted by the lower case letters, are defined as follows

$$c_t \equiv \frac{C_t}{e^{Z_t}}, \quad i_t \equiv \frac{I_t}{e^{Z_t}}, \quad k_t \equiv \frac{K_t}{e^{Z_t}}, \quad y_t \equiv \frac{Y_t}{e^{Z_t}}, \quad (B.33)$$

For notational ease in detrending the model, we define a combined shock $\varepsilon_{a,t}$, containing both the homoskedastic and heteroskedastic components of the productivity growth shock

$$\varepsilon_{a,t} \equiv \overline{\sigma_z} e^{\sigma_{z,t}} \varepsilon_{z,t} \quad (B.34)$$

The productivity growth process can therefore be written as

$$a_t \equiv Z_t - Z_{t-1} = \bar{a} + \varepsilon_{a,t} \quad (B.35)$$

While detrending, the exponential form of the foregoing will be frequently used

$$e^{a_t} = \frac{e^{Z_t}}{e^{Z_{t-1}}} = e^{\bar{a} + \varepsilon_{a,t}} \quad (B.36)$$

The goal is essentially to substitute C_t , I_t , K_t and Y_t for their stationary counterparts

⁴This begs the question, whether $y_{\sigma^2 \zeta}$ can ever be different from zero; i.e. whether stochastic volatility can ever have an effect through the risk channel. If $\rho \neq 0$, the conditional expectation of (B.25) at the deterministic steady state is

$$y_{\sigma^2 \eta} = - (f_{y^+} y_y + f_y)^{-1} \left(f_{y^+} y_{\sigma^2 \zeta} \rho + f_{y^+} y_{\varepsilon \omega^2 \zeta} E_t \left[\varepsilon_{t+1}^{\omega \otimes [2]} \right] + 2 f_{y^+ 2} (y_\omega \otimes y_{\omega \zeta}) E_t \left[\omega_{t+1}^{\otimes [2]} \right] \right) \tau \quad (B.31)$$

in the relevant model equations. We start with the production function

$$(y_t e^{Z_t}) = (k_{t-1} e^{Z_{t-1}})^\alpha (e^{Z_t} N_t)^{1-\alpha} \quad (\text{B.37})$$

$$\Rightarrow y_t = \left(\frac{e^{Z_t}}{e^{Z_{t-1}}} \right)^{-\alpha} k_{t-1}^\alpha N_t^{1-\alpha} \quad (\text{B.38})$$

$$\Rightarrow y_t = e^{-\alpha(\bar{a} + \varepsilon_{a,t})} k_{t-1}^\alpha N_t^{1-\alpha} \quad (\text{B.39})$$

Detrending the capital accumulation law

$$(k_t e^{Z_t}) = (1 - \delta) (k_{t-1} e^{Z_{t-1}}) + (i_t e^{Z_t}) \quad (\text{B.40})$$

$$\Rightarrow k_t = (1 - \delta) \frac{e^{Z_{t-1}}}{e^{Z_t}} k_{t-1} + i_t \quad (\text{B.41})$$

$$\Rightarrow k_t = (1 - \delta) e^{-\bar{a} - \varepsilon_{a,t}} k_{t-1} + i_t \quad (\text{B.42})$$

Detrending the market clearing condition is straightforward as it is a contemporaneous relationship

$$(y_t e^{Z_t}) = (c_t e^{Z_t}) + (i_t e^{Z_t}) \quad (\text{B.43})$$

$$\Rightarrow y_t = c_t + i_t \quad (\text{B.44})$$

Combing (B.39), (B.42) and (B.44) yields the detrended resource constraint

$$c_t + k_t = e^{-\alpha(\bar{a} + \varepsilon_{a,t})} k_{t-1}^\alpha N_t^{1-\alpha} + (1 - \delta) e^{-\bar{a} - \varepsilon_{a,t}} k_{t-1} \quad (\text{B.45})$$

Detrending the labor supply equation

$$\frac{\psi}{1 - N_t} = \frac{1}{c_t e^{Z_t}} (1 - \alpha) (k_{t-1} e^{Z_{t-1}})^\alpha e^{Z_t(1-\alpha)} N_t^{-\alpha} \quad (\text{B.46})$$

$$\Rightarrow \frac{\psi}{1 - N_t} = (1 - \alpha) e^{-\alpha(\bar{a} + \varepsilon_{a,t})} \frac{1}{c_t} k_{t-1}^\alpha N_t^{-\alpha} \quad (\text{B.47})$$

The risky rate r_t is stationary and we reexpress it in terms of the stationary variables

$$1 + r_t = (1 - \delta) + \alpha (k_{t-1} e^{Z_{t-1}})^{\alpha-1} (e^{Z_t} N_t)^{1-\alpha} \quad (\text{B.48})$$

$$\Rightarrow 1 + r_t = (1 - \delta) + \alpha k_{t-1}^{\alpha-1} e^{(\bar{a} + \varepsilon_{a,t})(1-\alpha)} N_t^{1-\alpha} \quad (\text{B.49})$$

We now move to the value function. As the felicity function is logarithmic in nonstationary consumption, removing the trend in consumption will leave a term linear in the level of productivity that when subtracted from V_t gives the stationary value function v_t

$$v_t = V_t - b \ln e^{Z_t} = V_t - b Z_t \quad (\text{B.50})$$

Substituting the relevant variables for their stationary counterparts yields

$$v_t + b Z_t = \ln(c_t e^{Z_t}) + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} [v_{t+1} + b Z_{t+1}] \right) \right] \right) \quad (\text{B.51})$$

$$\Rightarrow v_t = \ln c_t + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + b \left(Z_{t+1} - \frac{b-1}{b\beta} Z_t \right) \right] \right) \right] \right) \quad (\text{B.52})$$

It follows that the remaining nonstationarities can be offset if

$$\frac{b-1}{b\beta} = 1 \quad (\text{B.53})$$

which pins down b as

$$b = \frac{1}{1-\beta} \quad (\text{B.54})$$

Inserting (B.54) in (B.52) yields the stationary value function

$$v_t = \ln c_t + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \ln \left(E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right) \right] \right) \quad (\text{B.55})$$

While stationary, the foregoing value function does not fit in the problem statement (4.22) in the text, thus can not be implemented directly in perturbation software packages like Dynare. This problem is caused by nonlinear twisting of the expected continuation value, and can be fixed by redefining this conditional expectation as a new variable known in period t . Besides, the twisted expected continuation value is numerically unstable, due to the logarithmic transformation, when γ approaches zero or becomes very large. To counteract this, we define⁵

$$\tilde{v}_t \equiv E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} \tilde{\varepsilon}_{t+1} - \bar{v} \right] \right) \right] \quad (\text{B.56})$$

where \bar{v} denotes the deterministic steady state value of the stationary value function (B.55) and can be computed as follows

$$\bar{v} = \frac{1}{1-\beta} \left[\ln c + \psi \ln(1 - N) + \frac{\beta}{1-\beta} \bar{a} \right] \quad (\text{B.57})$$

Substituting v_{t+1} in (B.55) for \tilde{v}_t yields the normalized, stationary value function

$$v_t = \ln c_t + \psi \ln(1 - N_t) + \beta \frac{2}{\gamma} \left[\ln \tilde{v}_t + \frac{\gamma}{2} \left(\frac{1}{1-\beta} \bar{a} + \bar{v} \right) \right] \quad (\text{B.58})$$

With the stationary value function in hand, we reexpress the pricing kernel in terms of stationary variables

$$m_{t+1} = \beta \frac{c_t e^{Z_t}}{c_{t+1} e^{Z_{t+1}}} \frac{\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} Z_{t+1} \right] \right)}{E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} Z_{t+1} \right] \right) \right]} \quad (\text{B.59})$$

Multiplying both the denominator and numerator of the foregoing with $\exp \left(-\frac{\gamma}{2} \frac{1}{1-\beta} Z_t \right)$, and rearranging yields

$$m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\bar{a} + \varepsilon_{a,t+1})} \frac{\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right)}{E_t \left[\exp \left(\frac{\gamma}{2} \left[v_{t+1} + \frac{1}{1-\beta} (\bar{a} + \varepsilon_{a,t+1}) \right] \right) \right]} \quad (\text{B.60})$$

Writing out the definition of $\varepsilon_{a,t+1}$ yields (4.21) in the text. Recognizing the expecta-

⁵Rudebusch and Swanson (2012) adopt, in their companion Mathematica codes, a very similar procedure to improve numerical stability.

tional term in the previous equation can be replaced by the product $\tilde{v}_t \exp\left(\frac{\gamma}{2}\left[\bar{v} + \frac{1}{1-\beta}\bar{a}\right]\right)$, we substitute it for this product and collect terms

$$m_{t+1} = \beta \frac{c_t}{c_{t+1}} e^{-(\bar{a} + \varepsilon_{a,t+1})} \frac{\exp\left(\frac{\gamma}{2}\left[v_{t+1} + \frac{1}{1-\beta}\varepsilon_{a,t+1} - \bar{v}\right]\right)}{\tilde{v}_t} \quad (\text{B.61})$$

The period t counterpart of the foregoing follows

$$m_t = \beta \frac{c_{t-1}}{c_t} e^{-(\bar{a} + \varepsilon_{a,t})} \frac{\exp\left(\frac{\gamma}{2}\left[v_t + \frac{1}{1-\beta}\varepsilon_{a,t} - \bar{v}\right]\right)}{\tilde{v}_{t-1}} \quad (\text{B.62})$$

B.1.5 Proof of Proposition 4.6.1

Rearrange the definition of the second order increment to express the second order approximation as the sum of the first order approximation, the second order increment, and the second order constant risk adjustment

$$y_t^{(2)} = y_t^{(1)} + dy_t^{(2)} + \frac{1}{2}y\sigma^2 \quad (\text{B.63})$$

Applying the expectations operator to the foregoing yields the mean of the second order approximation

$$Ey_t^{(2)} = Ey_t^{(1)} + Eddy_t^{(2)} + \frac{1}{2}y\sigma^2 \quad (\text{B.64})$$

Centering the second order approximation (B.63) around its mean by subtracting (B.64) from (B.63) yields

$$y_t^{(2)} - Ey_t^{(2)} = \left(y_t^{(1)} - Ey_t^{(1)}\right) + \left(dy_t^{(2)} - Eddy_t^{(2)}\right) \quad (\text{B.65})$$

Noting that the mean of the first order approximation is the deterministic steady state of y_t , i.e., $Ey_t^{(1)} = \bar{y}$, the foregoing can be rewritten as

$$y_t^{(2)} - Ey_t^{(2)} = \left(y_t^{(1)} - \bar{y}\right) + \left(dy_t^{(2)} - Eddy_t^{(2)}\right) \quad (\text{B.66})$$

Using the definition of the first order increment $dy_t^{(1)} \equiv y_t^{(1)} - \bar{y}$, the foregoing is

$$y_t^{(2)} - Ey_t^{(2)} = dy_t^{(1)} + \left(dy_t^{(2)} - Eddy_t^{(2)}\right) \quad (\text{B.67})$$

Multiplying the foregoing with its transposition at $t-j$ and noting that $Ey_t^{(2)} = Ey_{t-j}^{(2)}$ and $Eddy_t^{(2)} = Eddy_{t-j}^{(2)}$ yields

$$\begin{aligned} & \left(y_t^{(2)} - Ey_t^{(2)}\right) \left(y_{t-j}^{(2)} - Ey_{t-j}^{(2)}\right)' \\ &= \left[dy_t^{(1)} + \left(dy_t^{(2)} - Eddy_t^{(2)}\right)\right] \left[dy_{t-j}^{(1)} + \left(dy_{t-j}^{(2)} - Eddy_{t-j}^{(2)}\right)\right]' \\ &= dy_t^{(1)} dy_{t-j}^{(1)'} + \left(dy_t^{(2)} dy_{t-j}^{(1)'} - Eddy_t^{(2)} dy_{t-j}^{(1)'}\right) \\ & \quad + \left(dy_t^{(1)} dy_{t-j}^{(2)'} - dy_t^{(1)} Eddy_{t-j}^{(2)'}\right) + \left(dy_t^{(2)} - Eddy_t^{(2)}\right) \left(dy_{t-j}^{(2)} - Eddy_{t-j}^{(2)}\right)' \end{aligned} \quad (\text{B.68})$$

Applying the expectations operator to the foregoing delivers

$$\begin{aligned} & E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \right] \\ &= E \left(dy_t^{(1)} dy_{t-j}^{(1)'} \right) + E \left(dy_t^{(2)} dy_{t-j}^{(1)'} \right) - E dy_t^{(2)} E dy_{t-j}^{(1)'} \\ & \quad + E \left(dy_t^{(1)} dy_{t-j}^{(2)'} \right) - E dy_t^{(1)} E dy_{t-j}^{(2)'} + E \left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_{t-j}^{(2)} \right)' \right] \end{aligned} \quad (\text{B.69})$$

To simplify the foregoing, apply the expectations operator to the definition of the first order increment, yielding its mean

$$E dy_t^{(1)} = E y_t^{(1)} - \bar{y} \quad (\text{B.70})$$

As $E y_t^{(1)} = \bar{y}$, the foregoing implies that the mean of the first order increment is zero

$$E dy_t^{(1)} = 0 \quad (\text{B.71})$$

Using this result and noting that $E dy_t^{(1)} = E dy_{t-j}^{(1)}$, (B.69) reduces to

$$\begin{aligned} & E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \right] \\ &= E \left(dy_t^{(1)} dy_{t-j}^{(1)'} \right) + E \left(dy_t^{(2)} dy_{t-j}^{(1)'} \right) + E \left(dy_t^{(1)} dy_{t-j}^{(2)'} \right) \\ & \quad + E \left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_{t-j}^{(2)} \right)' \right] \end{aligned} \quad (\text{B.72})$$

It then remains to show that

$$E \left(dy_t^{(2)} dy_{t-j}^{(1)'} \right) = 0, \quad E \left(dy_t^{(1)} dy_{t-j}^{(2)'} \right) = 0 \quad (\text{B.73})$$

One way is to use the moving average representation of the order increments. I.e., inserting the moving average representation of the first and second order approximations in the definition of the order increments yields

$$dy_t^{(1)} = \sum_{i=0}^{\infty} y_i \varepsilon_{t-i} \quad (\text{B.74})$$

$$dy_t^{(2)} = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{j,i} (\varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (\text{B.75})$$

Therefore the product of the two order increments, when set in expectation, takes the form of the third moments of the shocks, which is equal to zero under normality.

B.1.6 Proof of Proposition 4.6.2

First note that $E y_t^{(3)} = E y_t^{(2)}$ under normality⁶. Given this result, applying the expectations operator to the definition of the third order increment $dy_t^{(3)} \equiv y_t^{(3)} - y_t^{(2)}$

⁶To see this, applying the expectations operator to the second order approximation (4.30) and comparing the resulting expression with the mean of the third order approximation (4.25)

immediately implies $E dy_t^{(3)} = 0$.

Next, rearranging the definition of the third order increment delivers

$$y_t^{(3)} = y_t^{(2)} + dy_t^{(3)} \quad (\text{B.76})$$

Applying the expectations operator to the foregoing yields

$$E y_t^{(3)} = E y_t^{(2)} \quad (\text{B.77})$$

Centering (B.76) around its mean by subtracting (B.77) from (B.76) gives

$$y_t^{(3)} - E y_t^{(3)} = y_t^{(2)} - E y_t^{(2)} + dy_t^{(3)} \quad (\text{B.78})$$

Multiplying the foregoing with its transposition at $t - j$ and noting $E y_t^{(3)} = E y_{t-j}^{(3)}$ and $E y_t^{(2)} = E y_{t-j}^{(2)}$ delivers

$$\begin{aligned} \left(y_t^{(3)} - E y_t^{(3)} \right) \left(y_{t-j}^{(3)} - E y_{t-j}^{(3)} \right)' &= dy_t^{(3)} dy_{t-j}^{(3)'} + \left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \\ &\quad + dy_t^{(3)} y_{t-j}^{(2)'} - dy_t^{(3)} E y_{t-j}^{(2)'} + y_t^{(2)} dy_{t-j}^{(3)'} - E y_t^{(2)} dy_{t-j}^{(3)'} \end{aligned}$$

Applying the expectations operator to the foregoing, noting $E dy_t^{(3)} = 0$, gives

$$\begin{aligned} E \left[\left(y_t^{(3)} - E y_t^{(3)} \right) \left(y_{t-j}^{(3)} - E y_{t-j}^{(3)} \right)' \right] &= E_t \left[dy_t^{(3)} dy_{t-j}^{(3)'} \right] + E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \right] \\ &\quad + E \left(dy_t^{(3)} y_{t-j}^{(2)'} \right) + E \left(y_t^{(2)} dy_{t-j}^{(3)'} \right) \end{aligned} \quad (\text{B.79})$$

Rewrite the definition of the second order increment $dy_t^{(2)} \equiv y_t^{(2)} - y_t^{(1)} - \frac{1}{2} y_{\sigma^2}$ as

$$y_t^{(2)} = dy_t^{(2)} + y_t^{(1)} + \frac{1}{2} y_{\sigma^2} = dy_t^{(2)} + dy_t^{(1)} + \bar{y} + \frac{1}{2} y_{\sigma^2} \quad (\text{B.80})$$

Given the foregoing expression and noting $E dy_t^{(3)} = 0$, $E \left(y_t^{(2)} dy_{t-j}^{(3)'} \right)$ on the right hand side of (B.79) can be rewritten as

$$E \left(y_t^{(2)} dy_{t-j}^{(3)'} \right) = E \left[\left(dy_t^{(2)} + dy_t^{(1)} + \bar{y} + \frac{1}{2} y_{\sigma^2} \right) dy_{t-j}^{(3)'} \right] = E \left(dy_t^{(1)} dy_{t-j}^{(3)'} \right) \quad (\text{B.81})$$

Noting that $E \left(dy_t^{(2)} dy_{t-j}^{(3)'} \right)$ is zero under normality.⁷ Analogously, $E \left(dy_t^{(3)} y_{t-j}^{(2)'} \right)$ on the right hand side of (B.79) can be written as

$$E \left[dy_t^{(3)} y_{t-j}^{(2)'} \right] = E \left(dy_t^{(3)} dy_{t-j}^{(1)'} \right) \quad (\text{B.83})$$

⁷Again consider the moving average representation of the third order increment

$$dy_t^{(3)} = \frac{1}{2} \sum_{i=0}^{\infty} y_{\sigma^2, i} \varepsilon_{t-i} + \frac{1}{6} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{k, j, i} (\varepsilon_{t-k} \otimes \varepsilon_{t-j} \otimes \varepsilon_{t-i}) \quad (\text{B.82})$$

When multiplying with the moving average representation of the second order increment, the result, in expectation, is a sum of the third and fifth moments of shocks, and equal to zero under normality.

Inserting the last two equations in (B.79) yields

$$E \left[\left(y_t^{(3)} - E y_t^{(3)} \right) \left(y_{t-j}^{(3)} - E y_{t-j}^{(3)} \right)' \right] = E_t \left[d y_t^{(3)} d y_{t-j}^{(3)'} \right] + E \left[\left(y_t^{(2)} - E y_t^{(2)} \right) \left(y_{t-j}^{(2)} - E y_{t-j}^{(2)} \right)' \right] \\ + E \left(d y_t^{(1)} d y_{t-j}^{(3)'} \right) + E \left(d y_t^{(3)} d y_{t-j}^{(1)'} \right)$$

B.1.7 Mean of $dy_t^{(2)}$

The second order increment $dy_t^{(2)}$ can be expressed recursively as

$$dy_t^{(2)} = \alpha dy_{t-1}^{(2)state} + \frac{1}{2} \left[\beta_{22} dy_{t-1}^{(1)state \otimes [2]} + 2\beta_{20} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t \right) + \beta_{00} \varepsilon_t^{\otimes [2]} \right] \quad (B.84)$$

Taking expectation of the foregoing yields the following expression of the mean

$$E dy_t^{(2)} = \alpha E \left(dy_{t-1}^{(2)state} \right) + \frac{1}{2} \beta_{22} E \left(dy_{t-1}^{(1)state \otimes [2]} \right) + \frac{1}{2} \beta_{00} E \left(\varepsilon_t^{\otimes [2]} \right) \quad (B.85)$$

The two remaining unknown terms in the last equation, $E \left(dy_{t-1}^{(2)state} \right)$ and $E \left(dy_{t-1}^{(1)state \otimes [2]} \right)$, can be computed as follows. First, note the state variable block of (B.84) takes the form

$$dy_t^{(2)state} = \alpha^{state} dy_{t-1}^{(2)state} + \frac{1}{2} \beta_{22}^{state} dy_{t-1}^{(1)state \otimes [2]} + \beta_{20}^{state} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t \right) + \frac{1}{2} \beta_{00}^{state} \varepsilon_t^{\otimes [2]} \quad (B.86)$$

Taking expectation of the foregoing and rearranging yields the following expression for the mean of state variable block, noting throughout we use (ns) to denote the number of state variables

$$E dy_t^{(2)state} = (I_{ns} - \alpha^{state})^{-1} \left[\frac{1}{2} \beta_{22}^{state} E \left(dy_{t-1}^{(1)state \otimes [2]} \right) + \frac{1}{2} \beta_{00}^{state} E \left(\varepsilon_t^{\otimes [2]} \right) \right] \quad (B.87)$$

The problem now reduces to compute $E \left(dy_t^{(1)state \otimes [2]} \right)$. Once it is known, (B.87) gives the value of $E dy_t^{(2)state}$. Inserting these two values back in (B.85) yields the mean of the second order increment.

To compute $E \left(dy_t^{(1)state \otimes [2]} \right)$, we take the state variable block of the first order increment $dy_t^{(1)state}$ and raise it to the second Kronecker power,

$$dy_t^{(1)state \otimes [2]} = \alpha^{state \otimes [2]} dy_{t-1}^{(1)state \otimes [2]} + (K_{ns,ns} + I_{ns^2}) (\alpha^{state} \otimes \beta_0^{state}) \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t \right) \\ + \beta_0^{state \otimes [2]} \varepsilon_t^{\otimes [2]} \quad (B.88)$$

where $K_{ns,ns}$ is a $ns^2 \times ns^2$ commutation matrix (See Magnus and Neudecker (1979)).

Taking expectation of the foregoing and rearranging the resulting expression yields

$$E \left(dy_t^{(1)state \otimes [2]} \right) = \left(I_{ns^2} - \alpha^{state \otimes [2]} \right)^{-1} \beta_0^{state \otimes [2]} E \left(\varepsilon_t^{\otimes [2]} \right) \quad (B.89)$$

Inserting (B.89) and (B.87) back in (B.85) yields the mean of the second order

increment

$$E dy_t^{(2)} = \frac{1}{2} \left[\alpha (I_{ns} - \alpha^{state})^{-1} \beta_{00}^{state} + \left(\alpha (I_{ns} - \alpha^{state})^{-1} \beta_{22}^{state} + \beta_{22} \right) (I_{ns^2} - \alpha^{state \otimes [2]})^{-1} \beta_0^{state \otimes [2]} + \beta_{00} \right] E \left(\varepsilon_t^{\otimes [2]} \right) \quad (B.90)$$

which is an linear function of the second moments of the exogenous shocks. The coefficients on $E \left(\varepsilon_t^{\otimes [2]} \right)$ in the previous equation corresponds to the infinite sum $\sum_{j=0}^{\infty} y_{j,j}$, noting $y_{j,j} = \alpha y_{j-1,j-1}^{state} + \beta_{22} (y_{j-1}^{state} \otimes y_{j-1}^{state})$

$$\sum_{j=1}^{\infty} \alpha y_{j-1,j-1}^{state} = \alpha (I_{ns} - \alpha^{state})^{-1} \beta_{00}^{state} \quad (B.91)$$

$$\sum_{j=1}^{\infty} \beta_{22} (y_{j-1}^{state} \otimes y_{j-1}^{state}) = \left(\alpha (I_{ns} - \alpha^{state})^{-1} \beta_{22}^{state} + \beta_{22} \right) (I_{ns^2} - \alpha^{state \otimes [2]})^{-1} \beta_0^{state \otimes [2]} \quad (B.92)$$

$$y_{0,0} = \beta_{00} \quad (B.93)$$

B.1.8 Second Moments of $dy_t^{(2)}$

If (B.84) can be cast as a linear recursion, then standard linear methods can be applied to the computation of the second moments. Note $dy_t^{(2)}$, besides being linearly autoregressive in the state variable block of itself $dy_{t-1}^{(2)state}$, is a linear function of all the second order permutations of products of the first order increment $dy_{t-1}^{(1)state}$ and the shocks. This relationship guides the calculations, and we therefore compute the second moments of $dy_t^{(2)state}$ first, then recover the second moments of variables of interest.⁸

Combing (B.86) and (B.88) yields the following linear recursion containing the linear recursion of $dy_t^{(2)state}$

$$X_t^{(2)} = \Theta^{(2)X} X_{t-1}^{(2)} + \begin{bmatrix} \frac{1}{2} \beta_{00}^{state} \\ \beta_0^{state \otimes [2]} \end{bmatrix} E \left(\varepsilon_t^{\otimes [2]} \right) + \Phi^{(2)X} \Xi_t^{(2)} \quad (B.94)$$

⁸This procedure is widely adopted to minimize the dimension and improve the speed of the computation. See, e.g., Uhlig's (1999) toolkit, Schmitt-Grohé and Uribe's (2004) software package and Dynare.

where

$$X_t^{(2)} = \begin{bmatrix} dy_t^{(2)state} \\ dy_t^{(1)state \otimes [2]} \end{bmatrix} \quad (B.95)$$

$$\Theta^{(2)X} = \begin{bmatrix} \alpha^{state} & \frac{1}{2}\beta_{22}^{state} \\ 0 & \alpha^{state \otimes [2]} \end{bmatrix} \quad (B.96)$$

$$\Phi^{(2)X} = \begin{bmatrix} \frac{1}{2}\beta_{00}^{state} & \beta_{20}^{state} \\ \beta_0^{state \otimes [2]} & (K_{ns,ns} + I_{ns^2}) (\alpha^{state} \otimes \beta_0^{state}) \end{bmatrix} \quad (B.97)$$

$$\Xi_t^{(2)} = \begin{bmatrix} \varepsilon_t^{\otimes [2]} - E\varepsilon_t^{\otimes [2]} \\ dy_{t-1}^{(1)state} \otimes \varepsilon_t \end{bmatrix} \quad (B.98)$$

While the second term on the right hand side of (B.94) vanishes after centering (B.94) around its mean, it ensures, by compensating the subtraction of $E(\varepsilon_t^{\otimes [2]})$ in $\Xi_t^{(2)}$, that $\Xi_t^{(2)}$ is orthogonal⁹ to $X_{t-1}^{(2)}$

$$E(X_{t-1}^{(2)} \Xi_t^{(2)'}) = 0 \quad (B.99)$$

With the linear recursion of $X_t^{(2)}$, the second order increment (B.84) can be recast as the following linear recursion

$$dy_t^{(2)} = \Theta^{(2)}X_{t-1}^{(2)} + \frac{1}{2}\beta_{00}E(\varepsilon_t^{\otimes [2]}) + \Phi^{(2)}\Xi_t^{(2)} \quad (B.100)$$

$$\text{where } \Theta^{(2)} = \begin{bmatrix} \alpha & \frac{1}{2}\beta_{22} \end{bmatrix}, \quad \Phi^{(2)} = \begin{bmatrix} \frac{1}{2}\beta_{00} & \beta_{20} \end{bmatrix}$$

Noting $E(\Xi_t^{(2)}) = 0$ by construction. It follows that

$$E dy_t^{(2)} = \Theta^{(2)}EX_{t-1}^{(2)} + \frac{1}{2}\beta_{00}E(\varepsilon_t^{\otimes [2]}) \quad (B.101)$$

Contemporaneous Covariance

Centering (B.100) around its mean—by subtracting (B.101) from (B.100)—yields the following centered linear recursion of the second order increment

$$(dy_t^{(2)} - E dy_t^{(2)}) = \Theta^{(2)}(X_{t-1}^{(2)} - EX_{t-1}^{(2)}) + \Phi^{(2)}\Xi_t^{(2)} \quad (B.102)$$

Multiplying the foregoing with its transposition and applying the expectations operator to the resulting expression yields the contemporaneous variance of the second order increment

$$\Gamma_0^{(2)} = \Theta^{(2)}\Gamma_0^{(2)X}\Theta^{(2)'} + \Phi^{(2)}E(\Xi_t^{(2)}\Xi_t^{(2)'})\Phi^{(2)'} \quad (B.103)$$

⁹This orthogonality condition significantly simplifies the calculation of the autocovariances that followed.

where

$$\Gamma_0^{(2)X} = E \left[\left(X_t^{(2)} - EX_t^{(2)} \right) \left(X_t^{(2)} - EX_t^{(2)} \right)' \right] \quad (\text{B.104})$$

$$\Gamma_0^{(2)} = E \left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_t^{(2)} - E dy_t^{(2)} \right)' \right] \quad (\text{B.105})$$

This requires the contemporaneous variance of $X_t^{(2)}$, i.e., $\Gamma_0^{(2)X}$, as well as $E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right)$. Starting with $\Gamma_0^{(2)X}$, we can proceed by applying the expectations operator to (B.94) to yield

$$EX_t^{(2)} = \Theta^{(2)X} EX_t^{(2)} + \left[\frac{1}{2} \beta_{00}^{state} \right] E \left(\varepsilon_t^{\otimes [2]} \right) \quad (\text{B.106})$$

Centering the foregoing around its mean yields

$$X_t^{(2)} - EX_t^{(2)} = \Theta^{(2)X} \left(X_{t-1}^{(2)} - EX_{t-1}^{(2)} \right) + \Phi^{(2)X} \Xi_t^{(2)} \quad (\text{B.107})$$

Multiplying the foregoing with its transposition and applying the expectations operator, it follows the unknown contemporaneous variance of $X_t^{(2)}$ solves the following Lyapunov equation¹⁰

$$\Gamma_0^{(2)X} = \Theta^{(2)X} \Gamma_0^{(2)X} \Theta^{(2)X'} + \Phi^{(2)X} E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right) \Phi^{(2)X'} \quad (\text{B.108})$$

Thus, $\Gamma_0^{(2)X}$ can be calculated given $E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right)$ and, therefore, $\Gamma_0^{(2)X}$ in (B.103) too. We require this variance, which is given by

$$E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right) = \begin{bmatrix} (I_{ne^2} + K_{ne,ne}) [E(\varepsilon_t \varepsilon_t') \otimes E(\varepsilon_t \varepsilon_t')] & 0 \\ 0 & \Gamma_0^{(1)X} \otimes E(\varepsilon_t \varepsilon_t') \end{bmatrix} \quad (\text{B.109})$$

In the right hand side of (B.109), $\Gamma_0^{(1)X}$ is the state variable block of the contemporaneous variance of the first order approximation (or of the first order increment), and therefore already known from calculations at the first order.

The upper left entry of the right hand side of (B.109) contains the fourth moment of the shocks and has been simplified using Tracy and Sultan's (1993) formula. The two zero entries in (B.109) are due to the fact that the third moments of the shocks are zero under normality, and $dy_{t-1}^{(1)state}$ is uncorrelated with current shocks.

¹⁰Note $\Gamma_0^{(2)X}$ is of dimension $(ns + ns^2) \times (ns + ns^2)$. For models with a large number of state variables, splitting (B.108) into four Sylvester equations of smaller size by exploiting the triangularity of $\Theta^{(2)X}$ and solving them one by one is computationally a lot less expensive than solving (B.108) as a whole. This division also enables exploitation of the symmetry of $\Gamma_0^{(2)X}$ and therefore can avoid redundant computations.

Autocovariances

Now we turn to the autocovariances of $dy_t^{(2)}$. To start, note that under normality, $\Xi_t^{(2)}$ is serially uncorrelated

$$E \left(\Xi_t^{(2)} \Xi_{t-j}^{(2)'} \right) = 0 \quad \forall \quad j > 0 \quad (\text{B.110})$$

Given the contemporaneous variance $\Gamma_0^{(2)X}$, multiplying (B.107) with the transposition of (B.102) and taking expectation yields the contemporaneous variance between the $X_t^{(2)}$ and $dy_t^{(2)}$

$$\Gamma_0^{(2)X,dy} = \Theta^{(2)X} \Gamma_0^{(2)X} \Theta^{(2)'} + \Phi^{(2)X} E \left(\Xi_t^{(2)} \Xi_t^{(2)'} \right) \Phi^{(2)'} \quad (\text{B.111})$$

$$\text{where } \Gamma_0^{(2)X,dy} = E \left[\left(X_t^{(2)} - EX_t^{(2)} \right) \left(dy_t^{(2)} - E dy_t^{(2)} \right)' \right] \quad (\text{B.112})$$

With all the three contemporaneous variances in hand, the orthogonality (B.99) and (B.110) ensures the autocovariance of $dy_t^{(2)}$ can be computed with the following recursive formulae

$$\Gamma_j^{(2)} = \Theta^{(2)} \Gamma_{j-1}^{(2)X,dy} \quad (\text{B.113})$$

$$\Gamma_j^{(2)X,dy} = \Theta^{(2)X} \Gamma_{j-1}^{(2)X,dy} \quad (\text{B.114})$$

where

$$\Gamma_j^{(2)} = E \left[\left(dy_t^{(2)} - E dy_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_{t-j}^{(2)} \right)' \right] \quad (\text{B.115})$$

$$\Gamma_j^{(2)X,dy} = E \left[\left(X_t^{(2)} - EX_t^{(2)} \right) \left(dy_{t-j}^{(2)} - E dy_{t-j}^{(2)} \right)' \right] \quad (\text{B.116})$$

B.1.9 Second Moments of $dy_t^{(3)}$

The third order increment can be expressed recursively as

$$dy_t^{(3)} = \alpha dy_{t-1}^{(3)state} + \frac{1}{6} \left[\beta_{333,1} dy_{t-1}^{(1)state \otimes [3]} + \beta_{000} \varepsilon_t^{\otimes [3]} \right] \quad (\text{B.117})$$

$$\begin{aligned} & + \beta_{22} \left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state} \right) + \beta_{20} \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \\ & + \frac{1}{2} \left[\beta_{300} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes [2]} \right) + \beta_{330,1} \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) + \beta_{\sigma^2 0} \varepsilon_t + \beta_{\sigma^2 1} dy_{t-1}^{(1)state} \right] \end{aligned}$$

Its state variable block takes the form

$$\begin{aligned}
 dy_t^{(3)state} = & \alpha^{state} dy_{t-1}^{(3)state} + \frac{1}{6} \left[\beta_{333,1}^{state} dy_{t-1}^{(1)state \otimes [3]} + \beta_{000}^{state} \epsilon_t^{\otimes [3]} \right] \\
 & + \beta_{22}^{state} \left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state} \right) + \beta_{20}^{state} \left(dy_{t-1}^{(2)state} \otimes \epsilon_t \right) \\
 & + \frac{1}{2} \left[\beta_{300}^{state} \left(dy_{t-1}^{(1)state} \otimes \epsilon_t^{\otimes [2]} \right) + \beta_{330,1}^{state} \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \epsilon_t \right) \right. \\
 & \left. + \beta_{\sigma^2 0}^{state} \epsilon_t + \beta_{\sigma^2 1}^{state} dy_{t-1}^{(1)state} \right]
 \end{aligned} \tag{B.118}$$

From the terms on the left hand side of the foregoing, we need to build up two additional recursions, the first in the Kronecker product of the first and second order increments and the second in the triple Kronecker product of the first order increment, to construct the linear recursion containing $dy_t^{(3)state}$ that can be used for calculating moments

$$\begin{aligned}
 dy_t^{(2)state} \otimes dy_t^{(1)state} = & \alpha^{state \otimes [2]} \left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state} \right) + \left[\left(\frac{1}{2} \beta_{22}^{state} \right) \otimes \alpha^{state} \right] dy_{t-1}^{(1)state \otimes [3]} \\
 & + (\alpha^{state} \otimes \beta_0^{state}) \left(dy_{t-1}^{(2)state} \otimes \epsilon_t \right) + \left[\left(\frac{1}{2} \beta_{00}^{state} \right) \otimes \beta_0^{state} \right] \epsilon_t^{state \otimes [3]} \\
 & + \left[(\beta_{20}^{state} \otimes \alpha^{state}) K_{ns*ne,ns} + \left(\frac{1}{2} \beta_{22}^{state} \right) \otimes \beta_0^{state} \right] \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \epsilon_t \right) \\
 & + \left(\left[\left(\frac{1}{2} \beta_{00}^{state} \right) \otimes \alpha^{state} \right] K_{ne^2,ns} + \beta_{20}^{state} \otimes \beta_0^{state} \right) \left(dy_{t-1}^{(1)state} \otimes \epsilon_t^{\otimes [2]} \right) \\
 dy_t^{(1)state \otimes [3]} = & \alpha^{state \otimes [3]} dy_{t-1}^{(1)state \otimes [3]} + \beta_0^{state \otimes [3]} \epsilon_t^{\otimes [3]} \\
 & + [(K_{ns,ns} \otimes I_{ns} + I_{ns^3}) K_{ns^2,ns} + I_{ns^3}] \left(\alpha^{state \otimes [2]} \otimes \beta_0^{state} \right) \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \epsilon_t \right) \\
 & + [K_{ns^2,ns} + (K_{ns,ns} \otimes I_{ns} + I_{ns^3})] \left(\alpha^{state} \otimes \beta_0^{state \otimes [2]} \right) \left(dy_{t-1}^{(1)state} \otimes \epsilon_t^{\otimes [2]} \right)
 \end{aligned} \tag{B.119}$$

$$\begin{aligned}
 dy_t^{(1)state \otimes [3]} = & \alpha^{state \otimes [3]} dy_{t-1}^{(1)state \otimes [3]} + \beta_0^{state \otimes [3]} \epsilon_t^{\otimes [3]} \\
 & + [(K_{ns,ns} \otimes I_{ns} + I_{ns^3}) K_{ns^2,ns} + I_{ns^3}] \left(\alpha^{state \otimes [2]} \otimes \beta_0^{state} \right) \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \epsilon_t \right) \\
 & + [K_{ns^2,ns} + (K_{ns,ns} \otimes I_{ns} + I_{ns^3})] \left(\alpha^{state} \otimes \beta_0^{state \otimes [2]} \right) \left(dy_{t-1}^{(1)state} \otimes \epsilon_t^{\otimes [2]} \right)
 \end{aligned} \tag{B.120}$$

Given the foregoing two equations, along with the state variable block of the first order increment

$$dy_t^{(1)state} = \alpha^{state} dy_{t-1}^{(1)state} + \beta_0^{state} \epsilon_t \tag{B.121}$$

we construct the following linear recursion

$$X_t^{(3)} = \Theta^{(3)X} X_{t-1}^{(3)} + \Phi^{(3)X} \Xi_t^{(3)} \tag{B.122}$$

where¹¹

$$X_t^{(3)} = \begin{bmatrix} dy_t^{(3)state} \\ dy_t^{(2)state} \otimes dy_t^{(1)state} \\ dy_t^{(1)state \otimes [3]} \\ dy_t^{(1)state} \end{bmatrix}, \quad \Xi_t^{(3)} = \begin{bmatrix} \varepsilon_t^{\otimes [3]} \\ dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \\ dy_{t-1}^{(1)state} \otimes (\varepsilon_t^{\otimes [2]} - E \varepsilon_t^{\otimes [2]}) \\ dy_{t-1}^{(2)state} \otimes \varepsilon_t \\ \varepsilon_t \end{bmatrix} \quad (B.123)$$

Note there is no need to center $X_t^{(3)}$ before computing its contemporaneous variance as its mean is zero under normality, i.e., $E X_t^{(3)} = 0$. In the third entry of $\Xi_t^{(3)}$, $\varepsilon_t^{\otimes [2]}$ is adjusted using its mean, such that $\Xi_t^{(3)}$ is orthogonal to $X_{t-1}^{(3)}$

$$E \left(X_{t-1}^{(3)} \Xi_t^{(3)'} \right) = 0 \quad (B.124)$$

and it can be shown that $\Xi_t^{(3)}$ is serially uncorrelated

$$E \left(\Xi_t^{(3)} \Xi_{t-j}^{(3)'} \right) = 0 \quad \forall j > 0 \quad (B.125)$$

Contemporaneous Covariance

With linear recursion (B.122), the third order increment (B.117) can be cast in a linear recursion¹²

$$dy_t^{(3)} = \Theta^{(3)} X_{t-1}^{(3)} + \Phi^{(3)} \Xi_t^{(3)} \quad (B.126)$$

Multiplying the foregoing with its transposition and applying the expectations operator to the resulting expression yields the contemporaneous variance of the third order increment

$$\Gamma_0^{(3)} = \Theta^{(3)} \Gamma_0^{(3)X} \Theta^{(3)'} + \Phi^{(3)} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)'} \quad (B.127)$$

$$\text{where } \Gamma_0^{(3)} = E \left(dy_t^{(3)} dy_t^{(3)'} \right) \quad (B.128)$$

To compute the yet known contemporaneous variance of $X_t^{(3)}$, i.e., $\Gamma_0^{(3)X}$, we multiply (B.122) with its transposition and apply the expectations operator to the resulting expression. It follows that $\Gamma_0^{(3)X}$ solves the following Lyapunov equation¹³

$$\Gamma_0^{(3)X} = \Theta^{(3)X} \Gamma_0^{(3)X} \Theta^{(3)X'} + \Phi^{(3)X} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)X'} \quad (B.129)$$

$$\text{where } \Gamma_0^{(3)X} = E \left(X_t^{(3)} X_t^{(3)'} \right) \quad (B.130)$$

¹¹ $\Theta^{(3)X}$ and $\Phi^{(3)X}$ are specified in section B.1.12.

¹² $\Theta^{(3)}$ and $\Phi^{(3)}$ are specified in section B.1.12.

¹³ Note that (B.129) is a Lyapunov equation of dimension $(ns + ns^2 + ns^3 + ns) \times (ns + ns^2 + ns^3 + ns)$. By exploiting the triangularity of $\Theta^{(3)X}$ and the symmetry of $\Gamma_0^{(3)X}$, that large Lyapunov equation can be split and reduced to 10 Sylvester equations of dimension up to $ns^3 \times ns^3$.

with $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$ as specified in section B.1.12.

Given $\Gamma_0^{(3)X}$, multiplying (B.122) with the transposition of (B.126) and applying the expectations operator yields the contemporaneous variance between $X_t^{(3)}$ and $dy_t^{(3)}$

$$\Gamma_0^{(3)X,dy} = \Theta^{(3)X} \Gamma_0^{(3)X} \Theta^{(3)'} + \Phi^{(3)X} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)'} \quad (\text{B.131})$$

$$\text{where } \Gamma_0^{(3)X,dy} = E \left(X_t^{(3)} dy_t^{(3)'} \right) \quad (\text{B.132})$$

Autocovariances

For the autocovariance of the third order increment, the orthogonality (B.124) and $\Xi_t^{(3)}$ being serially uncorrelated, i.e., (B.125), ensure that it can be computed with the following recursive formulae

$$\Gamma_j^{(3)} = \Theta^{(3)} \Gamma_{j-1}^{(3)X,dy} \quad (\text{B.133})$$

$$\Gamma_j^{(3)X,dy} = \Theta^{(3)X} \Gamma_{j-1}^{(3)X,dy} \quad (\text{B.134})$$

where

$$\Gamma_j^{(3)} = E \left(dy_t^{(3)} dy_{t-j}^{(3)'} \right) \quad (\text{B.135})$$

$$\Gamma_j^{(3)X,dy} = E \left(X_t^{(3)} dy_{t-j}^{(3)'} \right) \quad (\text{B.136})$$

B.1.10 Second Moments between $dy_t^{(1)}$ and $dy_t^{(3)}$

First rewrite the linear recursion of the first order increment (4.26) using $X_t^{(3)}$

$$dy_t^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \alpha \end{bmatrix} X_{t-1}^{(3)} + \begin{bmatrix} 0 & 0 & 0 & 0 & \beta_0 \end{bmatrix} \Xi_t^{(3)} \quad (\text{B.137})$$

Multiplying the foregoing with the transposition of the linear recursion of the third order increment (B.126), and applying the expectations operator to the resulting expression yields the contemporaneous covariance between $dy_t^{(1)}$ and $dy_t^{(3)}$

$$\Gamma_0^{(1),(3)} = \begin{bmatrix} 0 & 0 & 0 & \alpha \end{bmatrix} \Gamma_0^{(3)X} \Theta^{(3)'} + \begin{bmatrix} 0 & 0 & 0 & 0 & \beta_0 \end{bmatrix} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)'} \quad (\text{B.138})$$

$$\text{where } \Gamma_0^{(1),(3)} = E \left(dy_t^{(1)} dy_t^{(3)'} \right) \quad (\text{B.139})$$

The autocovariance, $\Gamma_j^{(1),(3)}$, can be computed using the following recursive formula

$$\Gamma_j^{(1),(3)} = \begin{bmatrix} 0 & 0 & 0 & \alpha \end{bmatrix} \Gamma_{j-1}^{(3)X,dy} \quad (\text{B.140})$$

B.1.11 Variance Decomposition

The decomposition the variance of the third order approximation follows directly from the decomposition of the third order increment. Defining

$$dy_t^{(3)} \equiv dy_t^{(3)amp} + dy_t^{(3)risk} \quad (B.141)$$

Multiplying the foregoing with its transposition and applying the expectations operator, a variance decomposition immediately follows

$$\Gamma_0^{(3)} = \Gamma_0^{(3)amp} + \Gamma_0^{(3)risk} + \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' \quad (B.142)$$

where

$$\Gamma_0^{(3)amp} = E \left(dy_t^{(3)amp} dy_t^{(3)amp'} \right) \quad (B.143)$$

$$\Gamma_0^{(3)risk} = E \left(dy_t^{(3)risk} dy_t^{(3)risk'} \right) \quad (B.144)$$

$$\Gamma_0^{(3)amp,risk} = E \left(dy_t^{(3)amp} dy_t^{(3)risk'} \right) \quad (B.145)$$

Proposition (4.6.2) in the text implies the contemporaneous variance of the variables of interest takes the form

$$\Gamma_0^{y^{(3)}} = \Gamma_0^{y^{(2)}} + \Gamma_0^{(3)} + \Gamma_0^{(1),(3)} + \left(\Gamma_0^{(1),(3)} \right)' \quad (B.146)$$

Inserting the decomposed $\Gamma_0^{(3)}$, i.e., (B.142), in the previous equation yields the decomposition of the contemporaneous variance of the variables of interest

$$\begin{aligned} \Gamma_0^{y^{(3)}} = & \Gamma_0^{y^{(2)}} + \Gamma_0^{(3)amp} + \Gamma_0^{(3)risk} + \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' \\ & + \Gamma_0^{(1),(3)} + \left(\Gamma_0^{(1),(3)} \right)' \end{aligned} \quad (B.147)$$

Note the decomposition (B.147) is not yet complete as the cross-contemporaneous variance $\Gamma_0^{(1),(3)}$ can be further broken down into two parts¹⁴

$$\begin{aligned} \Gamma_0^{(1),(3)} = & E \left(dy_t^{(1)} dy_t^{(3)'} \right) \\ = & E \left[dy_t^{(1)} \left(dy_t^{(3)amp} + dy_t^{(3)risk} \right)' \right] \\ = & E \left(dy_t^{(1)} dy_t^{(3)amp'} \right) + E \left(dy_t^{(1)} dy_t^{(3)risk'} \right) \\ = & \Gamma_0^{(1)amp,(3)amp} + \Gamma_0^{(1)amp,(3)risk} \end{aligned} \quad (B.148)$$

¹⁴In (B.148), $\Gamma_0^{(1)amp,(3)amp}$ is used to denote $E \left(dy_t^{(1)} dy_t^{(3)amp'} \right)$ as there is only amplification effects in the first order increment $dy_t^{(1)}$.

Inserting the foregoing in (B.147) yields the complete variance decomposition

$$\begin{aligned}\Gamma_0^{y(3)} = & \Gamma_0^{y(2)} + \Gamma_0^{(3)amp} + \Gamma_0^{(3)risk} + \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' \\ & + \Gamma_0^{(1)amp,(3)amp} + \Gamma_0^{(1)amp,(3)risk} \\ & + \left(\Gamma_0^{(1)amp,(3)amp} + \Gamma_0^{(1)amp,(3)risk} \right)'\end{aligned}\quad (B.149)$$

Letting $\Gamma_0^{y(3)amp}$ collect the contribution from all amplification channels of all three orders, $\Gamma_0^{y(3)risk,amp}$ collects all interaction between amplification and time-varying risk adjustment channels and $\Gamma_0^{y(3)risk}$ collects the contribution from the time-varying risk adjustment channel

$$\Gamma_0^{y(3)amp} = \Gamma_0^{y(2)} + \Gamma_0^{(3)amp} + \Gamma_0^{(1)amp,(3)amp} + \left(\Gamma_0^{(1)amp,(3)amp} \right)' \quad (B.150)$$

$$\Gamma_0^{y(3)risk,amp} = \Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' + \Gamma_0^{(1)amp,(3)risk} + \left(\Gamma_0^{(1)amp,(3)risk} \right)' \quad (B.151)$$

$$\Gamma_0^{y(3)risk} = \Gamma_0^{(3)risk} \quad (B.152)$$

Inserting the foregoing in (B.149) yields (4.44) in the text. Note the first order amplification effect reported in Table 4.5 is included in (B.150). In particular, it is included in $\Gamma_0^{y(2)}$. As implied by proposition 4.6.1, the contemporaneous variance of the second order approximation takes the form

$$\Gamma_0^{y(2)} = \Gamma_0^{y(1)} + \Gamma_0^{(2)} \quad (B.153)$$

where $\Gamma_0^{y(1)}$ captures the first order amplification effect.

To compute the individual terms in (B.149), first note $dy_t^{(3)amp}$ collects all amplification effects and $dy_t^{(3)risk}$ collects the time-varying risk adjustment effect in the third order increment

$$dy_t^{(3)amp} = \alpha dy_{t-1}^{(3)amp,state} + \frac{1}{6} \left[\beta_{333,1} dy_{t-1}^{(1)state \otimes [3]} + \beta_{000} \varepsilon_t^{\otimes [3]} \right] \quad (B.154)$$

$$\begin{aligned}& + \beta_{22} \left(dy_{t-1}^{(2)state} \otimes dy_{t-1}^{(1)state} \right) + \beta_{20} \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \\ & + \frac{1}{2} \left[\beta_{300} \left(dy_{t-1}^{(1)state} \otimes \varepsilon_t^{\otimes [2]} \right) + \beta_{330,1} \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \right]\end{aligned}$$

$$dy_t^{(3)risk} = \alpha dy_{t-1}^{(3)risk,state} + \frac{1}{2} \beta_{\sigma^2 0} \varepsilon_t + \frac{1}{2} \beta_{\sigma^2 1} dy_{t-1}^{(1)state} \quad (B.155)$$

We start with constructing an auxiliary vector $X_{t-1}^{(3D)}$ for this decomposition

$$X_t^{(3D)} = \begin{bmatrix} dy_t^{(3)amp,state} \\ dy_t^{(3)risk,state} \\ dy_t^{(2)state} \otimes dy_t^{(1)state} \\ dy_t^{(1)state \otimes [3]} \\ dy_t^{(1)state} \end{bmatrix} \quad (B.156)$$

With the foregoing auxiliary vector, $dy_t^{(3)amp}$ and $dy_t^{(3)risk}$ can be cast as linear recursions

$$dy_t^{(3)amp} = \Theta^{(3)amp} X_{t-1}^{(3D)} + \Phi^{(3)amp} \Xi_t^{(3)} \quad (B.157)$$

$$dy_t^{(3)risk} = \Theta^{(3)risk} X_{t-1}^{(3D)} + \Phi^{(3)risk} \Xi_t^{(3)} \quad (B.158)$$

where

$$\Theta^{(3)amp} = \begin{bmatrix} \alpha & 0 & \beta_{22} & \frac{1}{6}\beta_{333,1} & \frac{1}{2}\beta_{300}^{state} \left(I_{ns} \otimes E \varepsilon_t^{\otimes [2]} \right) \end{bmatrix} \quad (B.159)$$

$$\Theta^{(3)risk} = \begin{bmatrix} 0 & \alpha & 0 & 0 & \frac{1}{2}\beta_{\sigma^2_1} \end{bmatrix} \quad (B.160)$$

$$\Phi^{(3)amp} = \begin{bmatrix} \frac{1}{6}\beta_{000} & \frac{1}{2}\beta_{330,1} & \frac{1}{2}\beta_{300} & \beta_{20} & 0 \end{bmatrix} \quad (B.161)$$

$$\Phi^{(3)risk} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2}\beta_{\sigma^2_0} \end{bmatrix} \quad (B.162)$$

Multiplying (B.157) with its transposition and applying the expectations operator yields the contemporaneous variance $\Gamma_0^{(3)amp}$, which collects the contribution of amplification channels to the total variance of the third order increment

$$\Gamma_0^{(3)amp} = \Theta^{(3)amp} E \left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right) \Theta^{(3)amp'} + \Phi^{(3)amp} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'} \quad (B.163)$$

where $E \left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right)$ can be computed using the following relationship

$$X_t^{(3)} = A^D X_t^{(3D)} \quad (B.164)$$

where

$$A^D = \begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (B.165)$$

therefore

$$E \left(X_{t-1}^{(3D)} X_{t-1}^{(3D)'} \right) = A^{D+} E \left(X_{t-1}^{(3)} X_{t-1}^{(3)'} \right) A^{D+'} = A^{D+} \Gamma_0^{(3)X} A^{D+'} \quad (B.166)$$

where A^{D+} denotes the Moore-Penrose inverse of A^D and $\Gamma_0^{(3)X}$ is already known. Then

$\Gamma_0^{(3)amp}$ can be computed using

$$\Gamma_0^{(3)amp} = \left(\Theta^{(3)amp} A^{D+} \right) \Gamma_0^{(3)X} \left(\Theta^{(3)amp} A^{D+} \right)' + \Phi^{(3)amp} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'} \quad (\text{B.167})$$

Likewise, the contemporaneous variance $\Gamma_0^{(3)risk}$ collects the contribution of the time-varying risk adjustment channel to the total variance of the third order increment, and can be computed using

$$\Gamma_0^{(3)risk} = \left(\Theta^{(3)risk} A^{D+} \right) \Gamma_0^{(3)X} \left(\Theta^{(3)risk} A^{D+} \right)' + \Phi^{(3)risk} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)risk'} \quad (\text{B.168})$$

$\Gamma_0^{(3)amp,risk}$ and its transposition collects the contribution of the interaction between the amplification and time-vary risk adjustment channels to the total variance of the third order increment, and can be computed using

$$\Gamma_0^{(3)amp,risk} + \left(\Gamma_0^{(3)amp,risk} \right)' = \Gamma_0^{(3)} - \Gamma_0^{(3)amp} - \Gamma_0^{(3)risk} \quad (\text{B.169})$$

To compute $\Gamma_0^{(1)amp,(3)amp}$, multiply (B.137) with the transposition of (B.157) and apply the expectations operator to the resulting expression to yield

$$\begin{aligned} \Gamma_0^{(1)amp,(3)amp} &= \begin{bmatrix} 0 & 0 & 0 & \alpha \end{bmatrix} \Gamma_0^{(3)X} \left(\Theta^{(3)amp} A^{D+} \right)' \\ &\quad + \begin{bmatrix} 0 & 0 & 0 & 0 & \beta_0 \end{bmatrix} E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) \Phi^{(3)amp'} \end{aligned} \quad (\text{B.170})$$

As $\Gamma_0^{(1),(3)}$ was already computed in section B.1.10, $\Gamma_0^{(1)amp,(3)risk}$ can be obtained by subtracting the foregoing from $\Gamma_0^{(1),(3)}$.

B.1.12 Coefficient Matrices

This section contains explicit expressions for several coefficient matrices left implicit above.

$$\Theta^{(3)} = \begin{bmatrix} \alpha & \beta_{22} & \frac{1}{6}\beta_{333,1} & \frac{1}{2}\beta_{300} \left(I_{ns} \otimes E \mathcal{E}_t^{\otimes [2]} \right) + \frac{1}{2}\beta_{\sigma^2 1} \end{bmatrix}$$

$$\Phi^{(3)} = \begin{bmatrix} \frac{1}{6}\beta_{000} & \frac{1}{2}\beta_{330,1} & \frac{1}{2}\beta_{300} & \beta_{20} & \frac{1}{2}\beta_{\sigma^2 0} \end{bmatrix}$$

$$\Theta^{(3)X} = \begin{bmatrix} \alpha^{state} & \beta_{22}^{state} & \frac{1}{6}\beta_{333,1}^{state} & \frac{1}{2} \left[\beta_{300}^{state} \left(I_{ns} \otimes E \varepsilon_t^{\otimes [2]} \right) + \beta_{\sigma^2 1}^{state} \right] \\ 0 & \alpha^{state \otimes [2]} & \left(\frac{1}{2} \beta_{22}^{state} \right) \otimes \alpha^{state} & \left(\left[\left(\frac{1}{2} \beta_{00}^{state} \right) \otimes \alpha^{state} \right] K_{ne^2, ns} + \beta_{20}^{state} \otimes \beta_0^{state} \right) \left(I_{ns} \otimes E \varepsilon_t^{\otimes [2]} \right) \\ 0 & 0 & \alpha^{state \otimes [3]} & \left[K_{ns^2, ns} + (K_{ns, ns} \otimes I_{ns} + I_{ns^3}) \right] \left(\alpha^{state} \otimes \beta_0^{state \otimes [2]} \right) \left(I_{ns} \otimes E \varepsilon_t^{\otimes [2]} \right) \\ 0 & 0 & 0 & \alpha^{state} \end{bmatrix}$$

$$\Phi^{(3)X} = \begin{bmatrix}
\frac{1}{6}\beta_{000}^{state} & \frac{1}{2}\beta_{330,1}^{state} & \dots \\
(\frac{1}{2}\beta_{00}^{state}) \otimes \beta_0^{state} & (\beta_{20}^{state} \otimes \alpha^{state}) K_{ns*ne,ns} + (\frac{1}{2}\beta_{22}^{state}) \otimes \beta_0^{state} & \dots \\
\beta_0^{state \otimes [3]} & [(K_{ns,ns} \otimes I_{ns} + I_{ns^3}) K_{ns^2,ns} + I_{ns^3}] \left(\alpha^{state \otimes [2]} \otimes \beta_0^{state} \right) & \dots \\
0 & 0 & \dots \\
\dots & \frac{1}{2}\beta_{300}^{state} & \beta_{20}^{state} & \frac{1}{2}\beta_{\sigma^2 0}^{state} \\
\dots & [(\frac{1}{2}\beta_{00}^{state}) \otimes \alpha^{state}] K_{ne^2,ns} + \beta_{20}^{state} \otimes \beta_0^{state} & \alpha^{state} \otimes \beta_0^{state} & 0 \\
\dots & [K_{ns^2,ns} + (K_{ns,ns} \otimes I_{ns} + I_{ns^3})] \left(\alpha^{state} \otimes \beta_0^{state \otimes [2]} \right) & 0 & 0 \\
\dots & 0 & 0 & \beta_0^{state}
\end{bmatrix}$$

$$\begin{aligned}
E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right) = & \begin{bmatrix} E \left(\varepsilon_t^{\otimes [3]} \varepsilon_t^{\otimes [3]'} \right) & E \left[\varepsilon_t^{\otimes [3]} \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right)' \right] & \dots \\ E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \varepsilon_t^{\otimes [3]'} \right] & E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right)' \right] & \dots \\ 0 & 0 & \dots \\ E \left[\left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \varepsilon_t^{\otimes [3]'} \right] & E \left[\left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right)' \right] & \dots \\ E \left(\varepsilon_t \varepsilon_t^{\otimes [3]'} \right) & E \left[\varepsilon_t \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right)' \right] & \dots \end{bmatrix} \\
& \dots \qquad \qquad \qquad 0 \qquad \qquad \dots \\
& \dots \qquad \qquad \qquad 0 \qquad \qquad \dots \\
& \dots E \left[\left(dy_{t-1}^{(1)state} \otimes \left(\varepsilon_t^{\otimes [2]} - E \varepsilon_t^{\otimes [2]} \right) \right) \left(dy_{t-1}^{(1)state} \otimes \left(\varepsilon_t^{\otimes [2]} - E \varepsilon_t^{\otimes [2]} \right) \right)' \right] \dots \\
& \dots \qquad \qquad \qquad 0 \qquad \qquad \dots \\
& \dots \qquad \qquad \qquad 0 \qquad \qquad \dots \\
& \dots \qquad \qquad \qquad \begin{bmatrix} E \left[\varepsilon_t^{\otimes [3]} \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right)' \right] & E \left(\varepsilon_t^{\otimes [3]} \varepsilon_t' \right) \\ E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right)' \right] & E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \varepsilon_t' \right] \\ 0 & 0 \\ E \left[\left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right)' \right] & E \left[\left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right) \varepsilon_t' \right] \\ E \left[\varepsilon_t \left(dy_{t-1}^{(2)state} \otimes \varepsilon_t \right)' \right] & E \left(\varepsilon_t \varepsilon_t' \right) \end{bmatrix}
\end{aligned}$$

B.1.13 Computing Elements in $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$

For every nonzero entry of $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$ in section B.1.12, the terms inside the expectations operator are either i) second, fourth, or sixth moments of the shocks, or ii) the product of these moments with the state variable block of the order increments, i.e., $dy_{t-1}^{(2)state}$ and $dy_{t-1}^{(1)state}$. The fourth and sixth moments of the shocks can be computed using Tracy and Sultan's (1993) formulae. E.g., for sixth moments in the form $E \left(\varepsilon_t^{\otimes[3]} \varepsilon_t^{\otimes[3]'} \right)$, applying the mixed Kronecker product rule yields

$$E \left(\varepsilon_t^{\otimes[3]} \varepsilon_t^{\otimes[3]'} \right) = E \left(\varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' \right) \quad (\text{B.171})$$

then Tracy and Sultan's (1993) Theorem 3 (repeated here) can be applied directly

$$\begin{aligned} E \left(\varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' \otimes \varepsilon_t \varepsilon_t' \right) &= \left[E \left(\varepsilon_t \varepsilon_t' \right) \right]^{\otimes[3]} \left[K + (K_{ne} \otimes K_{ne,ne}) \right. \\ &\quad \left. + (K_{ne,ne} \otimes K_{ne}) + K_{ne,ne^2} (K_{ne,ne} \otimes K_{ne}) \right] \\ &\quad + K \left([\text{vec} \left(E \left(\varepsilon_t \varepsilon_t' \right) \right) \text{vec}' \left(E \left(\varepsilon_t \varepsilon_t' \right) \right)] \otimes E \left(\varepsilon_t \varepsilon_t' \right) \right) K \end{aligned} \quad (\text{B.172})$$

where

$$K = I_{ne^3} + K_{ne,ne^2} + K_{ne^2,ne} \quad (\text{B.173})$$

is a sum of commutation matrices (See Magnus and Neudecker (1979)).

For the fourth moment in the form $E \left(\varepsilon_t^{\otimes[3]} \varepsilon_t' \right)$, Jinadasa and Tracy's (1986) formula (repeated here) can likewise be applied directly

$$\begin{aligned} E \left(\varepsilon_t^{\otimes[3]} \varepsilon_t' \right) &= E \left(\varepsilon_t \varepsilon_t' \right) \otimes \text{vec} \left(E \left(\varepsilon_t \varepsilon_t' \right) \right) + \text{vec} \left(E \left(\varepsilon_t \varepsilon_t' \right) \right) \otimes E \left(\varepsilon_t \varepsilon_t' \right) \\ &\quad + (I_{ne} \otimes K_{ne,ne}) \left[\text{vec} \left(E \left(\varepsilon_t \varepsilon_t' \right) \right) \otimes E \left(\varepsilon_t \varepsilon_t' \right) \right] \end{aligned} \quad (\text{B.174})$$

For the entries in the form of a product between the moments and the state variable block of order increments, use the property of the Kronecker product of column vectors and the mixed Kronecker product rule to rearrange until they are in the form of a (Kronecker) product of two clusters: one cluster contains the state variable block of the order increments only, and the other contains (the product of) shocks only. As all the order increments of the last period are uncorrelated with the current shocks, the expected value of the two clusters can be computed separately. E.g.

$$\begin{aligned} E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \varepsilon_t^{\otimes[3]'} \right] &= E \left[dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \otimes \varepsilon_t^{\otimes[3]'} \right] \\ &= E \left[dy_{t-1}^{(1)state \otimes [2]} \otimes \left(\varepsilon_t \otimes \varepsilon_t^{\otimes[3]'} \right) \right] \\ &= E \left[dy_{t-1}^{(1)state \otimes [2]} \otimes \left(\varepsilon_t \varepsilon_t^{\otimes[3]'} \right) \right] \\ &= E \left(dy_{t-1}^{(1)state \otimes [2]} \right) \otimes E \left(\varepsilon_t \varepsilon_t^{\otimes[3]'} \right) \end{aligned} \quad (\text{B.175})$$

where $E \left(dy_{t-1}^{(1)state \otimes [2]} \right)$ was computed in section B.1.8 and $E \left(\varepsilon_t \varepsilon_t^{\otimes [3]'} \right)$ can be computed using the transposed version of (B.174).

In fact, many nonzero entries in $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$ can be recycled from the calculations in section B.1.8 and therefore need not to be computed again. E.g., the block entry in the second row and second column of $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$ can be written as

$$E \left[\left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right) \left(dy_{t-1}^{(1)state \otimes [2]} \otimes \varepsilon_t \right)' \right] = E \left(dy_{t-1}^{(1)state \otimes [2]} dy_{t-1}^{(1)state \otimes [2]'} \right) \otimes E \left(\varepsilon_t \varepsilon_t' \right) \quad (B.176)$$

The first term on the right hand side of the foregoing can be recycled from $\Gamma_0^{(2)X}$ as the lower right entry (the block entry in the second row and second column) of $\Gamma_0^{(2)X}$ takes the form

$$\Gamma_{0,22}^{(2)X} = E \left[\left(dy_{t-1}^{(1)state \otimes [2]} - E dy_{t-1}^{(1)state \otimes [2]} \right) \left(dy_{t-1}^{(1)state \otimes [2]} - E dy_{t-1}^{(1)state \otimes [2]} \right)' \right] \quad (B.177)$$

$$= E \left(dy_{t-1}^{(1)state \otimes [2]} dy_{t-1}^{(1)state \otimes [2]'} \right) - E \left(dy_{t-1}^{(1)state \otimes [2]} \right) E \left(dy_{t-1}^{(1)state \otimes [2]} \right)' \quad (B.178)$$

therefore

$$E \left(dy_{t-1}^{(1)state \otimes [2]} dy_{t-1}^{(1)state \otimes [2]'} \right) = \Gamma_{0,22}^{(2)X} + E \left(dy_{t-1}^{(1)state \otimes [2]} \right) E \left(dy_{t-1}^{(1)state \otimes [2]} \right)' \quad (B.179)$$

Some entries of $E \left(\Xi_t^{(3)} \Xi_t^{(3)'} \right)$ are zero as they contain one or some of terms equal to zero under normality: the odd moments of the exogenous shocks, $E \left(dy_t^{(1)state} \right)$, $E \left(dy_t^{(1)state \otimes [3]} \right)$ and $E \left(dy_t^{(1)state \otimes [5]} \right)$.

Erklärung über genutzte Hilfsmittel

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Hong Lan